arXiv: arXiv:0000.0000

SUPPLEMENT TO "LIMIT THEORY FOR GEOMETRIC STATISTICS OF POINT PROCESSES HAVING FAST DECAY OF CORRELATIONS"

BY B. BŁASZCZYSZYN, D. YOGESHWARAN * AND J. E. YUKICH †

Inria/ENS, Paris; Indian Statistical Institute, Bangalore and Lehigh University, Bethlehem

This supplement contains various auxiliary facts needed in the proofs. These facts, some of which are of independent interest, may also be found in the arXiv version [3] of this paper.

1. Appendix.

1.1. Facts needed in the proof of fast decay of correlations of the ξ-weighted measures.

LEMMA 1.1. *Let f, g be two real valued, symmetric functions defined on $(\mathbb{R}^d)^k$ and $(\mathbb{R}^d)^l$ respectively. Let $F := \frac{1}{k!}\sum_{\mathbf{x}\in\mathcal{X}^{(k)}} f(\mathbf{x})$ and $G := \frac{1}{l!}\sum_{\mathbf{x}'\in\mathcal{X}^{(l)}} g(\mathbf{x}')$ be the corresponding U-statistics of order k and l respectively, on the input $\mathcal{X}\subset\mathbb{R}^d$. Then we have:*
(i) The product $F\,G$ is a sum of U-statistics of order not greater than $k+l$.
(ii) Let $\mathcal{A}$ be a fixed, finite subset of $\mathbb{R}^d$. The statistic $F_{\mathcal{A}} := \frac{1}{k!}\sum_{\mathbf{x}\in(\mathcal{X}\cup\mathcal{A})^{(k)}} f(\mathbf{x})$ is a sum of U-statistics of $\mathcal{X}$ of order not greater than k.

Proof. The two statements follow from symmetrizing the inner summands in the below representations

$$F\,G = \sum_{m=\max(k,l)}^{k+l} \sum_{\mathbf{z}\in\mathcal{X}^{(m)}} \frac{f(z_1,\dots,z_k)g(z_{m-l+1},\dots,z_m)}{(k+l-m)!(m-k)!(m-l)!},$$

$$F_{\mathcal{A}} = \sum_{m=0}^{\min(|\mathcal{A}|,k)} \sum_{\mathbf{a}\in\mathcal{A}^{(m)}} \sum_{\mathbf{z}\in\mathcal{X}^{(k-m)}} \frac{f(a_1,\dots,a_m,z_1,\dots,z_{k-m})}{m!(k-m)!},$$

*Research supported by DST-INSPIRE faculty award, CPDA from the Indian Statistical Institute and TOPOSYS grant.

†Research supported in part by NSF grant DMS-1406410

For a proof of the first representation note that by symmetry of f, $F(\mathcal{Y}) = f(y_1, \ldots, y_k)$ if $Y = \{y_1, \ldots, y_k\}$ and similarly for G. Thus, we derive that

$$\begin{aligned}
FG &= \sum_{\mathcal{Y}_i \subset \mathcal{X}, i=1,2} F(\mathcal{Y}_1)G(\mathcal{Y}_2)\mathbf{1}[|\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\
&= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathcal{Y}_i \subset \mathcal{X}, i=1,2} F(\mathcal{Y}_1)G(\mathcal{Y}_2)\mathbf{1}[|\mathcal{Y}_1 \cup \mathcal{Y}_2| = m, |\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\
&= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathcal{Y} \subset \mathcal{X}} \mathbf{1}[|\mathcal{Y}| = m] \sum_{\mathcal{Y}_i \subset \mathcal{Y}, i=1,2} F(\mathcal{Y}_1)G(\mathcal{Y}_2)\mathbf{1}[\mathcal{Y}_1 \cup \mathcal{Y}_2 = \mathcal{Y}, |\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\
&= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathbf{z} \in \mathcal{X}^{(m)}} \frac{1}{m!} \sum_{\mathcal{Y}_1, \mathcal{Y}_2} F(\mathcal{Y}_1)G(\mathcal{Y}_2)\mathbf{1}[\mathcal{Y}_1 \cup \mathcal{Y}_2 = \{z_1, \ldots, z_m\}, |\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\
&= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathbf{z} \in \mathcal{X}^{(m)}} \frac{f(z_1, \ldots, z_k)g(z_{m-l+1}, \ldots, z_m)}{m!} \\
&\qquad \times \sum_{\mathcal{Y}_1, \mathcal{Y}_2} \mathbf{1}[\mathcal{Y}_1 \cup \mathcal{Y}_2 = \{z_1, \ldots, z_m\}, |\mathcal{Y}_1| = k, |\mathcal{Y}_2| = l] \\
&= \sum_{m=\max(k,l)}^{k+l} \sum_{\mathbf{z} \in \mathcal{X}^{(m)}} \frac{f(z_1, \ldots, z_k)g(z_{m-l+1}, \ldots, z_m)}{(k+l-m)!(m-k)!(m-l)!},
\end{aligned}$$

thus proving the first representation above. The second representation follows similarly. □

LEMMA 1.2. *Let ξ be a score function on locally finite input $\mathcal{X}$ and $R^\xi := R^\xi(x, \mathcal{X})$ its radius of stabilization. Given $t > 0$ consider the score function $\tilde{\xi}(x, \mathcal{X}) := \xi(x, \mathcal{X})\mathbf{1}[R^\xi(x, \mathcal{X}) \leq t]$. Then the radius of stabilization $R^{\tilde{\xi}} := R^{\tilde{\xi}}(x, \mathcal{X})$ of $\tilde{\xi}$ is bounded by t, i.e., $R^{\tilde{\xi}}(x, \mathcal{X}) \leq t$ for any $x \in \mathcal{X}$.*

PROOF. Let $\mathcal{X}, \mathcal{A}$ be locally finite subsets of $\mathbb{R}^d$ with $x \in \mathcal{X}$. We have

$$\begin{aligned}
&\tilde{\xi}(x, (\mathcal{X} \cap B_t(x)) \cup (\mathcal{A} \cap B_t^c(x))) \\
&= \xi(x, (\mathcal{X} \cap B_t(x)) \cup (\mathcal{A} \cap B_t^c(x)))\, \mathbf{1}\left[R^\xi(x, (\mathcal{X} \cap B_t(x)) \cup (\mathcal{A} \cap B_t^c(x))) \leq t\right] \\
&= \xi(x, \mathcal{X} \cap B_t(x))\, \mathbf{1}\left[R^\xi(x, (\mathcal{X} \cap B_t(x)) \cup (\mathcal{A} \cap B_t^c(x))) \leq t\right],
\end{aligned}$$

where the last equality follows from the definition of R^ξ. Notice

$$\mathbf{1}\left[R^\xi(x, (\mathcal{X} \cap B_t(x)) \cup (\mathcal{A} \cap B_t^c(x))) \leq t\right] = \mathbf{1}\left[R^\xi(x, \mathcal{X} \cap B_t(x)) \leq t\right]$$

and so $\tilde{\xi}(x, (\mathcal{X} \cap B_t(x)) \cup (\mathcal{A} \cap B_t^c(x))) = \tilde{\xi}(x, \mathcal{X} \cap B_t(x))$, which was to be shown. □

1.2. **Determinantal and permanental point process lemmas.** The following facts illustrate the tractability of determinantal and permanental point processes and are of independent interest.We are indebted to Manjunath Krishnapur for sketching the proof of this result.

LEMMA 1.3. *Let $\mathcal{P}$ be a stationary determinantal point process on $\mathbb{R}^d$ with a kernel satisfying $K(x,y) \le \omega(|x-y|)$, where ω is a fast decreasing function. Then*

$$|\rho^{(n)}(x_1,\dots,x_{p+q}) - \rho^{(p)}(x_1,\dots,x_p)\rho^{(q)}(x_{p+1},\dots,x_{p+q})| \le n^{1+\frac{n}{2}}\omega(s)\|K\|^{n-1}, \tag{1.1}$$

where $\|K\| := \sup_{x,y\in\mathbb{R}^d}|K(x,y)|$, $s := s := d(\{x_1,\dots,x_p\},\{x_{p+1},\dots,x_{p+q}\}) := \inf_{i\in\{1,\dots,p\},j\in\{p+1,\dots,p+q\}}|x_i - x_j|$, *and* $n = p+q$.

Proof. Define the matrices $K_0 := (K(x_i,x_j))_{1\le i,j\le n}, K_1 := (K(x_i,x_j))_{1\le i,j\le p}$, and $K_2 := (K(x_i,x_j))_{p+1\le i,j\le n}$. Let L be the block diagonal matrix with blocks K_1, K_2. We define $\|K_0\| := \sup_{1\le i,j\le n}|K_0(x_i,x_j)|$ and similarly for the other matrices. Then

$$\begin{aligned}
|\rho^{(n)}(x_1,\dots,x_{p+q}) &- \rho^{(p)}(x_1,\dots,x_p)\rho^{(q)}(x_{p+1},\dots,x_{p+q})| \\
&= |\det(K_0) - \det(K_1)\det(K_2)| = |\det(K_0) - \det(L)| \\
&\le n^{1+\frac{n}{2}}\|K_0 - L\|\|K_0\|^{n-1} \le n^{1+\frac{n}{2}}\omega(s)\|K\|^{n-1} \qquad (1.2)
\end{aligned}$$

where the inequality follows by [1, (3.4.5)]. This gives (1.1). □

As a first step to prove the analogue of Lemma 1.3 for permanental point processes, we prove an analogue of (1.2). We follow verbatim the proof of (1.2) as given in [1, (3.4.5)]. Instead of using Hadamard's inequality for determinants as in [1], we use the following version of Hadamard's inequality for permanents ([4, Theorem 1.1]): For any column vectors $v_1,\dots,v_n$ of length n with complex entries, it holds that

$$|\mathrm{per}([v_1,\dots,v_n])| \le \frac{n!}{n^{\frac{n}{2}}}\prod_{i=1}^n \sqrt{\bar{v_i}^T v_i} \le n!\prod_{i=1}^n \|v_i\|,$$

where $\|v_i\|$ is the l_∞-norm of v_i viewed as an n-dimensional complex vector.

LEMMA 1.4. *Let $n\in\mathbb{N}$. For any two matrices K and L, we have*

$$|\mathrm{per}(K) - \mathrm{per}(L)| \le nn!\|K-L\|\max\{\|K\|,\|L\|\}^{n-1}.$$

Now, in the proof of Lemma 1.3, using the above estimate instead of (1.2), we establish the analogue for permanental point processes with fast-decreasing kernels K.

LEMMA 1.5. *Let $\mathcal{P}$ be a stationary permanental point process on $\mathbb{R}^d$ with a fast-decreasing kernel satisfying $K(x,y) \leq \omega(|x-y|)$ where ω is a fast decreasing function. With s as in Lemma 1.3 and $n = p+q$, we have*

$$|\rho^{(n)}(x_1,\ldots,x_{p+q}) - \rho^{(p)}(x_1,\ldots,x_p)\rho^{(q)}(x_{p+1},\ldots,x_{p+q})| \leq nn!\omega(s)\|K\|^{n-1}.$$

To bound the radius of stabilization of geometric functionals on determinantal point processes, we rely on the following exponential decay of Palm void probabilities. Though the proof is inspired by that of a similar estimate in [7, Lemma 2], we derive a more general and explicit bound.

LEMMA 1.6. *Let $\mathcal{P}$ be a stationary determinantal point process on $\mathbb{R}^d$. Then for $p, k \in \mathbb{N}$, $\mathbf{x} \in (\mathbb{R}^d)^p$, and any bounded Borel subset $B \subset \mathbb{R}^d$, we have*

$$\mathbb{P}^!_{\mathbf{x}}(\mathcal{P}(B) \leq k) \leq e^{(2k+p)/8} e^{-K(\mathbf{0},\mathbf{0})\mathrm{Vol}_d(B)/8}. \tag{1.3}$$

Proof. For any determinantal point process $\mathcal{P}$ (even a non-stationary one), let $\mathcal{P}_x$ be the reduced Palm point process with respect to $x \in \mathbb{R}^d$. From [9, Theorem 6.5], we have that $\mathcal{P}_x$ is also a determinantal point process and its kernel L is given by

$$L(y_1,y_2) = K(y_1,y_2) - \frac{K(y_1,x)K(x,y_2)}{K(x,x)}. \tag{1.4}$$

Next we assert that

$$\int_{\mathbb{R}^d} |K(x,y)|^2 \mathrm{d}y \leq K(x,x), \; x \in \mathbb{R}^d. \tag{1.5}$$

To see this, write $K(x,y) = \sum_j \lambda_j \phi_j(x)\bar{\phi}_j(y)$, $\lambda_j \in [0,1]$, where $\phi_j, j \geq 1$, is an orthonormal basis for $L^2(\mathbb{R}^d, dx)$ (cf. Lemma 4.2.2 of [2]). In view of $\overline{K(x,y)} = K(y,x)$ we get $\int_{\mathbb{R}^d} |K(x,y)|^2 \mathrm{d}y = \int_{\mathbb{R}^d} K(x,y)K(y,x)\mathrm{d}y = \sum_j \lambda_j^2 \phi_j(x)\bar{\phi}_j(x) \leq \sum_j \lambda_j \phi_j(x)\bar{\phi}_j(x) = K(x,x)$, whence the assertion (1.5). The bound (1.5) shows for any bounded Borel subset B and $x \in \mathbb{R}^d$ that

$$\mathbb{E}^!_x(\mathcal{P}(B)) = \int_B L(y,y)\mathrm{d}y = \int_B K(y,y)\mathrm{d}y - \frac{1}{K(x,x)} \int_B |K(x,y)|^2 \mathrm{d}y \geq \mathbb{E}(\mathcal{P}(B)) - 1.$$

Re-iterating the above inequality, we get that for all $\mathbf{x} \in \mathbb{R}^{dp}$ and any bounded Borel subset B

$$\mathbb{E}^!_{\mathbf{x}}(\mathcal{P}(B)) \geq \mathbb{E}(\mathcal{P}(B)) - p. \tag{1.6}$$

Since the point count of a determinantal point process in a given set is a sum of independent Bernoulli random variables [2, Theorem 4.5.3], the Chernoff-Hoeffding bound [6, Theorem 4.5] yields

$$\mathbb{P}^!_{\mathbf{x}}(\mathcal{P}(B) \leq \mathbb{E}^!_{\mathbf{x}}(\mathcal{P}(B))/2) \leq e^{-\mathbb{E}^!_{\mathbf{x}}(\mathcal{P}(B))/8}. \tag{1.7}$$

Now we return to our stationary determinantal point process $\mathcal{P}$ and note that $\mathbb{E}(\mathcal{P}(B)) = K(\mathbf{0},\mathbf{0})\mathrm{Vol}_d(B)$. Suppose first that B is large enough so that $K(\mathbf{0},\mathbf{0})\mathrm{Vol}_d(B) \geq 2k+p$. Thus combining (1.6) and (1.7), we have

$$\mathbb{P}^!_{\mathbf{x}}(\mathcal{P}(B) \leq k) \leq \mathbb{P}^!_{\mathbf{x}}(\mathcal{P}(B) \leq \mathbb{E}^!_{\mathbf{x}}(\mathcal{P}(B))/2) \leq e^{-(K(\mathbf{0},\mathbf{0})\mathrm{Vol}_d(B)-p)/8}.$$

On the other hand, if B is small and satisfies $K(\mathbf{0},\mathbf{0})\mathrm{Vol}_d(B_{r_0}) < 2k+p$, then the right-hand side of (1.10) is larger than 1 and hence it is a trivial bound. $\square$

Inequality (1.6) can also be deduced from the stronger coupling result of [8, Prop. 5.10(iv)] for determinantal point processes with a continuous kernel but we have given an elementary proof. Given Ginibre input, we may improve the exponent in the void probability bound (1.10). We believe this result to be of independent interest, as it generalizes [10, Lemma 6.1], which treats the case $k = 0$.

LEMMA 1.7. *Let $B_r := B_r(\mathbf{0}) \subset \mathbb{R}^2$ and $\mathcal{P}$ be the Ginibre point process. Then for $p, k \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^{2p}$,*

$$\begin{aligned}\mathbb{P}^!_{\mathbf{x}}(\mathcal{P}(B_r) \leq k) &\leq \exp\{p(k+1)r^2\}\mathbb{P}(\mathcal{P}(B_r) \leq k) \\ &\leq kr^{2k}\exp\{(p(k+1)+k)r^2 - \frac{1}{4}r^4(1+o(1))\}.\end{aligned} \tag{1.8}$$

We remark that stationarity shows the above bound holds for any radius r ball.

Proof. We shall prove the result for $p = 1$ and use induction to deduce the general case.

Let $\mathcal{K}_{B_r}$ be the restriction to B_r of the integral operator $\mathcal{K}$ (generated by kernel K) corresponding to Ginibre point process and $\mathcal{L}_{B_r}$ be the restriction to B_r of the integral operator $\mathcal{L}$ (generated by kernel L) corresponding to the reduced Palm point process (also a determinantal point process). Let $\lambda_i, i = 1, 2, \ldots$ and $\mu_i, i = 1, 2, \ldots$ be the eigenvalues of $\mathcal{K}_{B_r}$ and $\mathcal{L}_{B_r}$ in decreasing order respectively.

Then from (1.4) we have that the rank of the operator $\mathcal{K}_{B_r} - \mathcal{L}_{B_r}$ is one. Secondly, note that

$$\sum_i \mu_i = \mathbb{E}_x(\mathcal{P}(B_r)) = \int_{B_r} L(y,y)\mathrm{d}y \leq \int_{B_r} K(y,y)\mathrm{d}y = \mathbb{E}(\mathcal{P}(B_r)) = \sum_i \lambda_i.$$

Hence, by a generalisation of Cauchy's interlacing theorem [5, Theorem 4] combined with the above inequality, we get the interlacing inequality $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for $i = 1, 2, \ldots$.

Now, fix $\mathbf{x} = x \in \mathbb{R}^2$. Again by [2, Theorem 4.5.3], we have that $P(B_r) \stackrel{d}{=} \sum_i \mathrm{Bernoulli}(\lambda_i)$ and under Palm measure, $P(B_r) \stackrel{d}{=} \sum_i \mathrm{Bernoulli}(\mu_i)$ where both the sums involve independent Bernoulli random variables. Independence of the Bernoulli random variables gives

$$
\begin{aligned}
\mathbb{P}_x(\mathcal{P}(B_r) \leq k) &= \sum_{J \subset \mathbb{N}, |J| \leq k} \prod_{j \in J} \mu_j \prod_{j \notin J} (1 - \mu_j) \\
&\leq \sum_{J \subset \mathbb{N}, |J| \leq k} \prod_{j \in J} \lambda_j \prod_{j \notin J} (1 - \lambda_{j+1}) \\
&\leq \sum_{J \subset \mathbb{N}, |J| \leq k} \prod_{j \in J} \lambda_j \prod_{j \notin J} (1 - \lambda_j) \prod_{j-1 \in J \cup \{0\}, j \notin J} (1 - \lambda_j)^{-1} \\
&\leq (1 - \lambda_1)^{-k-1} \sum_{J \subset \mathbb{N}, |J| \leq k} \prod_{j \in J} \lambda_j \prod_{j \notin J} (1 - \lambda_j) \\
&= (1 - \lambda_1)^{-k-1} \mathbb{P}[\mathcal{P}(B_r) \leq k].
\end{aligned}
$$

The proof of the first inequality in (1.8) for the case $p = 1$ is complete by noting that $\lambda_1 = \mathbb{P}(EXP(1) \leq r^2)$ (see [2, Theorems 4.7.1 and 4.7.3]), where $EXP(1)$ stands for an exponential random variable with mean 1. As said before, iteratively the first inequality in (1.8) can be proven for an arbitrary p. To complete the proof of the second inequality, we bound $\mathbb{P}(\mathcal{P}(B_r) \leq k)$ in a manner similar to the proof of [2, Proposition 7.2.1].

Let $\mathcal{P}^* := \{R_1^2, R_2^2, \ldots, \} = \{|X|^2 : X \in \mathcal{P}\}$ be the point process of squared modulii of the Ginibre point process. Then, from [2, Theorem 4.7.3], it is known that $R_i^2 \stackrel{d}{=} \Gamma(i, 1)$ ($\Gamma(i, 1)$ denotes a gamma random variable with parameters $i, 1$) and are independently distributed. There is a constant $\beta \in (0, 1)$ such that

$$
\mathbb{P}(R_i^2 \geq r^2) \leq e^{-\beta r^2} \mathbb{E}(e^{\beta R_i^2}) \leq e^{-\beta r^2} (1 - \beta)^{-i}, \; i \geq 1.
$$

For $i < r^2$, the bound is optimal for $\beta = 1 - \frac{i}{r^2}$. For r, set $r_* := \ulcorner r^2 \urcorner$, the ceiling of r^2. Then,

$$\begin{aligned}
\mathbb{P}(\mathcal{P}(B_r) \leq k) &= \mathbb{P}(\sharp\{i : R_i^2 \leq r^2\} \leq k) \leq \mathbb{P}(\sharp\{i \leq r_* : R_i^2 \leq r^2\} \leq k) \\
&\leq \sum_{J \subset [r_*], |J| \leq k} \prod_{i \in J} \mathbb{P}(R_j^2 \leq r^2) \prod_{i \notin J} \mathbb{P}(R_j^2 > r^2) \\
&\leq \sum_{J \subset [r_*], |J| \leq k} \prod_{i \in J} e^{r^2} e^{-\beta r^2} (1-\beta)^{-i} \prod_{i \notin J} e^{-\beta r^2} (1-\beta)^{-i} \\
&\leq kr^{2k} e^{kr^2} \prod_{i=1}^{r_*} e^{-ar^2} (1-\beta)^{-i} = kr^{2k} e^{kr^2} e^{-\frac{1}{4} r^4 (1+o(1))},
\end{aligned}$$

where equality follows by substituting the optimal β for each i, as in [2, Section 7.2]. □

1.3. **Facts about superposition of independent point processes.** The following facts on superposition of independent point processes were useful in the applications involving α-determinantal point processes, $|\alpha| = 1/m, m \in \mathbb{N}$. First recall that, for any $k \geq 1$ and distinct $x_1, \ldots, x_k \in \mathbb{R}^d$ the following relation holds

$$\rho_0^{(k)}(x_1, \ldots, x_k) = \sum_{\sqcup_{i=1}^m S_i = [k]} \prod_{i=1}^m \rho(S_i), \tag{1.9}$$

where $\sqcup$ stands for disjoint union and where we abbreviate $\rho^{(|S_i|)}(x_j : j \in S_i)$ by $\rho(S_i)$. Here S_i may be empty, in which case we set $\rho(\emptyset) = 1$. The proof of the next proposition, which shows that $\mathcal{P}_0$ has fast decay of correlations, is very useful for our purposes.

PROPOSITION 1.8. *Let $m \in \mathbb{N}$ and $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be i.i.d. copies of an admissible point process $\mathcal{P}$ having fast decay of correlations with decay function ϕ and correlation decay constants C_k and c_k. Then $\mathcal{P}_0 := \cup_{i=1}^m \mathcal{P}_i$ is an admissible point process having fast decay of correlations with decay function ϕ and correlation decay constants $m^k m! (\kappa_k)^{m-1} C_k$ and c_k. Further, if $\mathcal{P}$ is admissible input of type (A2) with $\kappa_k \leq \lambda^k$ for some $\lambda \in (0, \infty)$, then $\mathcal{P}_0$ is also admissible input of type (A2).*

Proof of Proposition 1.8. We shall prove the proposition in the case $m = 2$; the general case follows in the same fashion albeit with considerably more notation. Let $x_1, \ldots, x_{p+q}$ be distinct points in $\mathbb{R}^d$ with s at Lemma 1.3 as usual. For a subset

$S \subset [p+q]$, we abbreviate $\rho^{|S|}(x_j : j \in S)$ by $\rho(S)$. Using (1.9) we have that

$$\begin{aligned}
& \rho_0^{(p+q)}([p+q]) = \sum_{S_1 \sqcup S_2 = [p+q]} \rho(S_1)\rho(S_2) = 2\rho([p+q]) + 2\rho([p])\rho([q]) \\
& + \sum_{S_1 \sqcup S_2 = [p+q], S_2 \cap [p] = \emptyset, S_i \neq \emptyset} \rho(S_1)\rho(S_2) + \sum_{S_1 \sqcup S_2 = [p+q], S_1 \cap [p] = \emptyset, S_i \neq \emptyset} \rho(S_1)\rho(S_2) \\
& + \sum_{S_1 \sqcup S_2 = [p+q], S_2 \cap [q] = \emptyset, S_i \neq \emptyset} \rho(S_1)\rho(S_2) + \sum_{S_1 \sqcup S_2 = [p+q], S_1 \cap [q] = \emptyset, S_i \neq \emptyset} \rho(S_1)\rho(S_2) \\
& + \sum_{S_1 \sqcup S_2 = [p+q], S_i \cap [p] \neq \emptyset, S_i \cap [q] \neq \emptyset} \rho(S_1)\rho(S_2) \\
= \; & 2\rho([p+q]) + 2\rho([p])\rho([q]) \\
& + \sum_{S_{21} \sqcup S_{22} = [q], S_{ij} \neq \emptyset} (\rho(S_{21} \cup [p])\rho(S_{22}) + \rho(S_{22} \cup [p])\rho(S_{21})) \\
& + \sum_{S_{11} \sqcup S_{12} = [p], S_{ij} \neq \emptyset} (\rho(S_{11} \cup [q])\rho(S_{12}) + \rho(S_{12} \cup [q])\rho(S_{11})) \\
& + \sum_{S_{21} \sqcup S_{22} = [q], S_{11} \sqcup S_{12} = [p], S_{ij} \neq \emptyset} \rho(S_{11} \cup S_{21})\rho(S_{12} \cup S_{22}).
\end{aligned}$$

On the other hand the product of correlation functions is

$$\begin{aligned}
& \rho_0([p])\rho_0([q]) = (\sum_{S_{11} \sqcup S_{12} = [p]} \rho(S_{11})\rho(S_{12}))(\sum_{S_{21} \sqcup S_{22} = [q]} \rho(S_{21})\rho(S_{22})) \\
= \; & (2\rho([p]) + \sum_{S_{11} \sqcup S_{12} = [p], S_{ij} \neq \emptyset} \rho(S_{11})\rho(S_{12})) \times (2\rho([q]) + \sum_{S_{21} \sqcup S_{22} = [q], S_{ij} \neq \emptyset} \rho(S_{21})\rho(S_{22})) \\
= \; & 4\rho([p])\rho([q]) + 2 \sum_{S_{21} \sqcup S_{22} = [q], S_{ij} \neq \emptyset} \rho(S_{21})\rho([p])\rho(S_{22}) \\
& +2 \sum_{S_{11} \sqcup S_{12} = [p], S_{ij} \neq \emptyset} \rho(S_{11})\rho([q])\rho(S_{12}) \\
& + \sum_{S_{21} \sqcup S_{22} = [q], S_{11} \sqcup S_{12} = [p], S_{ij} \neq \emptyset} \rho(S_{11})\rho(S_{21})\rho(S_{12})\rho(S_{22}).
\end{aligned}$$

Now, we shall match the two summations term-wise and bound the differences using the bound on correlation functions and fast decay of correlations condition :

$$\begin{aligned}
& |\rho_0([p+q]) - \rho_0([p])\rho_0([q])| \leq 2|\rho([p+q]) - \rho([p])\rho([q])| \\
& + \sum_{S_{21} \sqcup S_{22} = [q], S_{ij} \neq \emptyset} |\rho(S_{21} \cup [p])\rho(S_{22}) - \rho(S_{21})\rho([p])\rho(S_{22})|
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{S_{21} \sqcup S_{22} = [q], S_{ij} \neq \emptyset} |\rho(S_{22} \cup [p])\rho(S_{21}) - \rho(S_{21})\rho([p])\rho(S_{22})| \\
&+ \sum_{S_{11} \sqcup S_{12} = [p], S_{ij} \neq \emptyset} |\rho(S_{11} \cup [q])\rho(S_{12}) - \rho(S_{11})\rho([q])\rho(S_{12})| \\
&+ \sum_{S_{11} \sqcup S_{12} = [p], S_{ij} \neq \emptyset} |\rho(S_{12} \cup [q])\rho(S_{11}) - \rho(S_{11})\rho([q])\rho(S_{12})| \\
&+ \sum_{S_{21} \sqcup S_{22} = [q], S_{11} \sqcup S_{12} = [p], S_{ij} \neq \emptyset} |\rho(S_{11} \cup S_{21})\rho(S_{12} \cup S_{22}) - \rho(S_{11})\rho(S_{21})\rho(S_{12})\rho(S_{22})| \\
\leq \; & 2\kappa_{p+q} C_{p+q} \phi(c_{p+q} s) \sum_{S_1 \sqcup S_2 = [p+q]} 1 \;=\; 2\kappa_{p+q} C_{p+q} \phi(c_{p+q} s) 2^{p+q}. \quad \square
\end{aligned}
$$

We now provide void probability bounds for superposition of independent point processes.

PROPOSITION 1.9. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_m, m \in \mathbb{N}$, be independent admissible point processes. For $p, k \in \mathbb{N}$ and a bounded Borel set B, set*

$$\nu_{p,k}(B) := \sup_{i=1,\ldots,m} \sup_{0 \leq p' \leq p} \sup_{x_1,\ldots,x_{p'}} \mathbb{P}_{x_1,\ldots,x_{p'}}(\mathcal{P}_i(B) \leq k).$$

Let $\mathcal{P} := \cup_{i=1}^m \mathcal{P}_i$ be the independent superposition. Then, $\alpha^{(k)}$ a.e. $x_1, \ldots, x_p$, we have

$$\mathbb{P}_{x_1,\ldots,x_p}(\mathcal{P}(B) \leq k) \leq \nu_{p,k}(B)^m.$$

PROOF. We shall show the proposition for $m = 2$ and the general case follows similarly. Further, we use ρ, ρ_1, ρ_2 to denote the correlation functions of $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ respectively. Let $A = A_1 \times \ldots \times A_p$ where $A_1, \ldots, A_p$ are disjoint bounded Borel subsets. By setting $f(x_1, \ldots, x_p; \mathcal{P}) = \mathbf{1}[\mathcal{P}(B) \leq k]$ in the refined Campbell theorem and using the independence of $\mathcal{P}_1, \mathcal{P}_2$, we derive that

$$
\begin{aligned}
&\int_A \mathbb{P}_{x_1,\ldots,x_p}(\mathcal{P}(B) \leq k)\rho^{(p)}(x_1, \ldots, x_p) \mathrm{d}x_1 \ldots \mathrm{d}x_p = \mathbb{E}(\mathbf{1}[\mathcal{P}(B) \leq k]\mathcal{P}(A_1) \ldots \mathcal{P}(A_p)) \\
&\leq \sum_{S \subset [p]} \mathbb{E}\left(\mathbf{1}[\mathcal{P}_1(B) \leq k] \prod_{i \in S} \mathcal{P}_1(A_i)\right) \mathbb{E}\left(\mathbf{1}[\mathcal{P}_2(B) \leq k] \prod_{i \notin S} \mathcal{P}_2(A_i)\right) \\
&= \int_A \sum_{S \subset [p]} \mathbb{P}_{x_i; i \in S}(\mathcal{P}_1(B) \leq k) \rho_1^{(|S|)}(x_i; i \in S) \mathbb{P}_{x_i; i \notin S}(\mathcal{P}_2(B) \leq k) \\
&\qquad \times \rho_2^{(p-|S|)}(x_i; i \notin S) \mathrm{d}x_1 \ldots \mathrm{d}x_p
\end{aligned}
$$

Thus, by definition of $\nu_{p,k}(B)$, we get that for a.e. $x_1, \ldots, x_p$

$$\begin{aligned}
&\mathbb{P}_{x_1,\ldots,x_p}(\mathcal{P}(B) \le k)\rho^{(p)}(x_1,\ldots,x_p) \\
&\le \sum_{S \subset [p]} \mathbb{P}_{x_i; i \in S}(\mathcal{P}_1(B) \le k)\rho_1^{(|S|)}(x_i; i \in S)\mathbb{P}_{x_i; i \notin S}(\mathcal{P}_2(B) \le k)\rho_2^{(p-|S|)}(x_i; i \notin S) \\
&\le \nu_{p,k}(B)^2 \sum_{S \subset [p]} \rho_1^{(|S|)}(x_i; i \in S)\rho_2^{(p-|S|)}(x_i; i \notin S) = \nu_{p,k}(B)^2 \rho^{(p)}(x_1,\ldots,x_p),
\end{aligned}$$

where the last equality follows from (1.9). The proposition now follows from the above inequality. $\square$

Now, as a trivial corollary of Lemma 1.6 and Proposition 1.9, we obtain the following useful result.

COROLLARY 1.10. *Let $\mathcal{P}$ be a stationary α-determinantal point process on $\mathbb{R}^d$ with $\alpha = -1/m, m \in \mathbb{N}$. Then for $p, k \in \mathbb{N}$, $\mathbf{x} \in (\mathbb{R}^d)^p$, and any bounded Borel subset $B \subset \mathbb{R}^d$, we have*

$$\mathbb{P}^!_{\mathbf{x}}(\mathcal{P}(B) \le k) \le e^{m(2k+p)/8} e^{-K(\mathbf{0},\mathbf{0})\mathrm{Vol}_d(B)/8}. \tag{1.10}$$

Consider the same assumptions as in Proposition 1.9. For a bounded Borel subset B, set

$$M_p(B) := \sup_{i=1,\ldots,m} \sup_{0 \le p' \le p} \sup_{x_1,\ldots,x_{p'}} \mathbb{E}_{x_1,\ldots,x_{p'}}(t^{\mathcal{P}_i(B)}), \ \ t \ge 0, p \ge 1.$$

Now setting $f(x_1,\ldots,x_p;\mathcal{P}) = t^{\mathcal{P}(B)}$ in the proof of the proposition, we may deduce that

$$\sup_{0 \le p' \le p} \sup_{x_1,\ldots,x_{p'}} \mathbb{E}_{x_1,\ldots,x_{p'}}(t^{\mathcal{P}(B)}) \le M_p(B)^m. \tag{1.11}$$

References.

[1] G. W. Anderson, A. Guionnet and O. Zeitouni (2010), *An Introduction to Random Matrices,* Cambridge University Press, Cambridge, U.K.

[2] J. Ben Hough, M. Krishnapur, Y. Peres, and B. Virág (2009), *Zeros of Gaussian Analytic Functions and Determinantal Point Processes,* Vol. 51, American Mathematical Society, Providence, RI.

[3] B. Błaszczyszyn, D. Yogeshwaran, and J. E. Yukich (2018), Limit theory for geometric statistics of point processes having fast decay of correlations, arXiv:1606.03988v3.

[4] E. Carlen, E. Lieb, M. Loss (2006), An inequality of Hadamard type for permanents, *Meth. Appl. Anal.*, **13**, 1, 1-18.

[5] J. Dancis and C. Davis (1987), An interlacing theorem for eigenvalues of self-adjoint operators, *Lin. Alg. Appl.*, **88–89**, 117–122.

[6] M. Mitzenmacher and E. Upfal (2005), *Probability and Computing Randomized Algorithms and Probabilistic Analysis,* Cambridge University Press, Cambridge.

[7] N. Miyoshi and T. Shirai. (2016), A sufficient condition for tail asymptotics of SIR distribution in downlink cellular networks. In *14th Int. Symp. on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt)*, 1–7.

[8] R. Pemantle and Y. Peres (2014), Concentration of Lipschitz functionals of determinantal and other strong Rayleigh measures, *Comb. Probab. Comput.*, **23**, 140–160.

[9] T. Shirai and Y. Takahashi (2003), Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes, *J. Funct. Anal.*, **205**, 2, 414–463.

[10] D. Yogeshwaran and R. J. Adler (2015), On the topology of random complexes built over stationary point processes, *Ann. Appl. Prob.*, **25**, No. 6, 3338–3380.

INRIA/ENS, 2 RUE SIMONE IFF
CS 42112 75589 PARIS CEDEX 12.
FRANCE.
E-MAIL: Bartek.Blaszczyszyn@ens.fr

STATISTICS AND MATHEMATICS UNIT
INDIAN STATISTICAL INSTITUTE
BANGALORE - 560059
E-MAIL: d.yogesh@isibang.ac.in

DEPARTMENT OF MATHEMATICS
LEHIGH UNIVERSITY
BETHLEHEM PA 18015
E-MAIL: joseph.yukich@lehigh.edu