Probab. Theory Relat. Fields (2014) 158:435–463
DOI 10.1007/s00440-013-0484-1

Variance asymptotics for random polytopes in smooth convex bodies

Pierre Calka · J. E. Yukich

Received: 19 June 2012 / Revised: 10 January 2013 / Published online: 9 February 2013

Abstract Let $K \subset \mathbb{R}^d$ be a smooth convex set and let $\mathcal{P}_\lambda$ be a Poisson point process on $\mathbb{R}^d$ of intensity λ. The convex hull of $\mathcal{P}_\lambda \cap K$ is a random convex polytope K_λ. As $\lambda \to \infty$, we show that the variance of the number of k-dimensional faces of K_λ, when properly scaled, converges to a scalar multiple of the affine surface area of K. Similar asymptotics hold for the variance of the number of k-dimensional faces for the convex hull of a binomial process in K.

Keywords Random polytopes · Affine surface area · Parabolic growth and hull processes

Mathematics Subject Classification (2000) Primary 60F05; Secondary 60D05

1 Introduction

Let $K \subset \mathbb{R}^d$ be a compact convex body with non-empty interior and having a C^3 boundary of positive Gaussian curvature κ. Letting $\mathcal{P}_\lambda$ be a Poisson point process

P. Calka was partially supported by French ANR grant PRESAGE. J. E. Yukich was supported in part by NSF grant DMS-1106619.

P. Calka
Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, Avenue de l'Université,
BP.12, Technopôle du Madrillet, 76801 Saint-Etienne-du-Rouvray, France
e-mail: pierre.calka@univ-rouen.fr

J. E. Yukich (✉)
Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA
e-mail: jey0@lehigh.edu

in $\mathbb{R}^d$ of intensity λ, $\lambda \in (0, \infty)$, we denote by K_λ the convex hull of $\mathcal{P}_\lambda \cap K$. Let $f_k(K_\lambda)$, $k \in \{0, 1, \dots, d-1\}$, be the number of k faces of K_λ.

Rényi and Sulanke [16] were the first to consider the average behavior of $f_0(K_\lambda)$ in the planar case. Generalizing their formula to higher dimensions, Bárány [1] showed there is a constant $D_{0,d}$ such that

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \mathbb{E} f_0(K_\lambda) = D_{0,d} \int_{\partial K} \kappa(z)^{1/(d+1)} dz.$$

The integral $\int_{\partial K} \kappa(z)^{1/(d+1)} dz$ is known as the *affine surface area* of the boundary ∂K. Assuming only that ∂K is of differentiability class C^2, Reitzner [15] extended this result to $f_k(K_\lambda)$, $k \in \{0, 1, \dots, d-1\}$, showing for all $d \geq 2$ that there are constants $D_{k,d}$ such that

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \mathbb{E} f_k(K_\lambda) = D_{k,d} \int_{\partial K} \kappa(z)^{1/(d+1)} dz. \tag{1.1}$$

Reitzner [14] also showed that $(f_k(K_\lambda) - \mathbb{E} f_k(K_\lambda))/\sqrt{\mathrm{Var} f_k(K_\lambda)}$ converges in distribution to a mean zero normal random variable as $\lambda \to \infty$, though there have been relatively few results concerning the asymptotic variance of $f_k(K_\lambda)$. Theorem 4 of Reitzner [14] gives upper and lower bounds of the same magnitude for $\mathrm{Var} f_k(K_\lambda)$, $k \in \{0, 1, \dots, d-1\}$, which extends work of Buchta [7], who obtains lower bounds for $\mathrm{Var} f_0(K_\lambda)$ of order $\lambda^{(d-1)/(d+1)}$. In the special case that K is a ball, closed form variance asymptotics for $\mathrm{Var} f_k(K_\lambda)$, $k \in \{0, 1, \dots, d-1\}$ are given in [8,19].

Let K'_n be the convex hull of n i.i.d. random variables uniformly distributed on K. Our main two results resolve the open question of determining variance asymptotics for $\mathrm{Var} f_k(K_\lambda)$ and $\mathrm{Var} f_k(K'_n)$, K smooth and convex, as put forth on p. 1431 of [21]. For all $m = 1, 2, \dots$, let $\mathcal{K}^m_+$ be the collection of compact convex sets K in $\mathbb{R}^d$ with boundary of class C^m and having positive Gaussian curvature.

Theorem 1.1 *For all $k \in \{0, 1, \dots, d-1\}$, there exists a constant $F_{k,d} \in (0, \infty)$ such that for all $K \in \mathcal{K}^3_+$*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \mathrm{Var} f_k(K_\lambda) = F_{k,d} \int_{\partial K} \kappa(z)^{1/(d+1)} dz. \tag{1.2}$$

Let vol denote Lebesgue measure on $\mathbb{R}^d$. De-Poissonization methods, based on coupling, yield the following binomial counterpart of (1.2). When $k = 0$, it resolves Conjecture 1 of Buchta [7].

Theorem 1.2 *For all $k \in \{0, 1, \dots, d-1\}$ and $K \in \mathcal{K}^3_+$*

$$\lim_{n\to\infty} n^{-(d-1)/(d+1)} \mathrm{Var} f_k(K'_n) = F_{k,d} (\mathrm{vol}(K))^{-(d-1)/(d+1)} \int_{\partial K} \kappa(z)^{1/(d+1)} dz. \tag{1.3}$$

Remarks (i) *Related work.* Bárány and Reitzner [4, p. 3] conjecture for general convex bodies that $\mathrm{Var} f_k(K_\lambda)$ should, up to constants, behave like the variance of the volume of the wet part of the floating body, which, in the case of smooth convex sets, is proportional to affine surface area. Theorem 1.1 resolves a sharpened version of this conjecture in the case that ∂K is smooth.

(ii) *The constants $F_{k,d}$.* The proofs of Theorems 1.1 and 1.2 show that $F_{k,d}$ is defined in terms of parabolic growth processes on $\mathbb{R}^{d-1} \times \mathbb{R}^+$, a fact first recognized by Tomasz Schreiber in the context of $K = \mathbb{B}^d$. As noted on p. 137 of Buchta [7], $F_{k,2}$ may also be identified in terms of a constant involving complicated double integrals given in Groeneboom [9].

Theorems 1.1 and 1.2 yield asymptotics for $\mathrm{Var}\ \mathrm{vol}(K'_n)$ and $\mathrm{Var}\ \mathrm{vol}(K_\lambda)$, which goes as follows. Recall that under C^3 and C^2 assumptions on ∂K, respectively, Bárány [1] and Reitzner [13] show

$$\lim_{\lambda\to\infty} \lambda^{2/(d+1)} \mathbb{E}\mathrm{vol}(K \setminus K_\lambda) = c_d (\mathrm{vol}(K))^{2/(d+1)} \int_{\partial K} \kappa(z)^{1/(d+1)} dz. \tag{1.4}$$

Böröczky et al. [6] extend this limit and (1.1) to convex hulls of i.i.d. points having a non-uniform density on K. Theorem 3 of Reitzner's breakthrough paper [14] gives upper and lower bounds of the same magnitude for $\mathrm{Var}\ \mathrm{vol}(K_\lambda)$, though it falls short of prescribing a limiting variance. Theorems 1.1 and 1.2 fill in this gap as follows. When $K \in \mathcal{K}_+^{d+6}$, Buchta notes (see Corollary 1 and (3.6) of [7]) that variance asymptotics for $n^2 \mathrm{Var} f_0(K'_n)$ and $\mathrm{Var}\ \mathrm{vol}(K'_n)$ coincide, since

$$\mathrm{Var}\ \mathrm{vol}(K'_n) = \frac{\mathrm{Var}(f_0(K'_{n+2})) + d_{n+2}}{(n+1)(n+2)},$$

where d_n, $n \geq 1$, satisfies

$$\lim_{n\to\infty} \left(\frac{3-d}{d+1} \int_{\partial K} \kappa(z)^{1/(d+1)} dz \cdot n^{(d-1)/(d+1)} \right)^{-1} d_n = 1.$$

Consequently, putting $k = 0$ in (1.3) and $G_d := F_{0,d} + (3-d)/(d+1)$ we get the following corollary.

Corollary 1.1 *For all $K \in \mathcal{K}_+^{d+6}$ we have*

$$\lim_{n\to\infty} n^{(d+3)/(d+1)} \mathrm{Var}\ \mathrm{vol}(K'_n) = G_d (\mathrm{vol}(K))^{(d+3)/(d+1)} \int_{\partial K} \kappa(z)^{1/(d+1)} dz. \tag{1.5}$$

By (1.5) and Proposition 3.2 of [20], which states that $\mathrm{Var}\ \mathrm{vol}(K'_n)$ and $\mathrm{Var}\ \mathrm{vol}(K_n)$ coincide up to first order, we deduce the next result.

Corollary 1.2 *For all $K \in \mathcal{K}_+^{d+6}$ we have*

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \text{Var vol}(K_\lambda) = G_d \int_{\partial K} \kappa(z)^{1/(d+1)} dz. \tag{1.6}$$

The paper is organized as follows. Section 2 introduces the main tool for the proof of Theorem 1.1, namely the paraboloid growth process used in [19] and [8]. We state a general result, Theorem 2.1, giving expectation and variance asymptotics for the empirical k-face measure, which includes Theorem 1.1 as a special case. Theorem 2.1 also shows that the constants $F_{k,d}$ of Theorem 1.1 involve integrals of one and two point correlation functions of a scaling limit k-face functional $\xi_k^{(\infty)}$ associated with parabolic growth processes. Section 3 introduces *an affine transform of K and a scaling transform of the affine transform* to link the finite volume k-face functional with the infinite volume scaling limit $\xi_k^{(\infty)}$. Section 4 contains the main technical aspects of the paper, focussing on properties of re-scaled k-face functionals. In particular Lemmas 4.5 and 4.8 show that the one and two point correlation functions of the re-scaled k-face functional on the affine transform of K are well approximated by one and two point correlation functions of the re-scaled k-face functional on an osculating ball. In this way the expectation and variance asymptotics for $f_k(K_\lambda)$, K an arbitrary smooth body, are controlled by the corresponding asymptotics for $f_k(K_\lambda)$ when K is a ball. The latter asymptotics are established in [8]. Section 5 contains the proof of Theorem 2.1 and Section 6 establishes the de-Poissonized limit (1.3).

2 Paraboloid growth processes and a general result

Given a finite point set $\mathcal{X} \subset \mathbb{R}^d$, let $\text{co}(\mathcal{X})$ be its convex hull.

Definition 2.1 Given $k \in \{0, 1, \dots, d-1\}$ and x a vertex of $\text{co}(\mathcal{X})$, define the k-face functional $\xi_k(x, \mathcal{X})$ to be the product of $(k+1)^{-1}$ and the number of k faces of $\text{co}(\mathcal{X})$ which contain x. Otherwise we put $\xi_k(x, \mathcal{X}) = 0$. The empirical k-face measure is

$$\mu_\lambda^{\xi_k} := \sum_{x \in \mathcal{P}_\lambda \cap K} \xi_k(x, \mathcal{P}_\lambda \cap K)\delta_x, \tag{2.1}$$

where δ_x is the unit point mass at x.

Thus the number of k-faces in $\text{co}(\mathcal{X})$ is $\sum_{x\in\mathcal{X}} \xi_k(x, \mathcal{X})$. We shall give a general result describing the limit behavior of $\mu_\lambda^{\xi_k}$ in terms of parabolic growth processes on $\mathbb{R}^d$.

Paraboloid growth processes. Denote points in $\mathbb{R}^{d-1} \times \mathbb{R}$ by $w := (v, h)$ or $w' := (v', h')$, depending on context. Let $\Pi^\uparrow$ be the epigraph of the parabola $v \mapsto |v|^2/2$, that is $\Pi^\uparrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}^+, h \geq |v|^2/2\}$. Letting $\mathcal{X} \subset \mathbb{R}^d$ be locally finite, define the parabolic growth model

$$\Psi(\mathcal{X}) := \bigcup_{w\in\mathcal{X}} (w \oplus \Pi^\uparrow),$$

where $\oplus$ denotes Minkowski addition. A point $w_0 \in \mathcal{X}$ is *extreme* with respect to $\Psi(\mathcal{X})$ if the epigraph $w_0 \oplus \Pi^{\uparrow}$ is not a subset of the union of the epigraphs $\{w \oplus \Pi^{\uparrow}, w \in \mathcal{X} \setminus w_0\}$, that is $(w_0 \oplus \Pi^{\uparrow}) \not\subseteq \bigcup_{w \in \mathcal{X} \setminus w_0} (w \oplus \Pi^{\uparrow})$.

The paraboloid hull model $\Phi(\mathcal{X})$ is defined as in Definition 3.4 of [8]:

$$\Phi(\mathcal{X}) := \bigcup_{\begin{cases} w \in \mathbb{R}^{d-1} \times \mathbb{R} \\ (w \oplus \Pi^{\downarrow}) \cap \mathcal{X} = \emptyset \end{cases}} (w \oplus \Pi^{\downarrow}),$$

where $\Pi^{\downarrow} := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}, h \leq -|v|^2/2\}$. It may be viewed as the dual of the paraboloid growth model $\Psi(\mathcal{X})$. Let $\mathcal{P}$ be a rate one homogeneous Poisson point process on $\mathbb{R}^{d-1} \times \mathbb{R}^+$ and let $\Psi := \Psi(\mathcal{P})$ and $\Phi := \Phi(\mathcal{P})$ be the corresponding *paraboloid growth and hull processes*. As in [8], the set Vertices(Φ) coincides with the extreme points of Ψ. We recall (cf. Definition 3.3 of [8]) that a set of $(k+1)$ extreme points $x_1, \ldots, x_{k+1}$, $k \geq 0$, generates a so-called k-dimensional *paraboloid face* if there exists a translate $\tilde{\Pi}^{\downarrow}$ of $\Pi^{\downarrow}$ such that $\{x_1, \ldots, x_{k+1}\} = \tilde{\Pi}^{\downarrow} \cap \mathcal{X}$.

Definition 2.2 *(cf. section 6 of [8])* Define the scaling limit k-face functional $\xi_k^{(\infty)}(x, \mathcal{P})$, for $x \in \mathcal{P}$, and $k \in \{0, 1, \ldots, d-1\}$, to be the product of $(k+1)^{-1}$ and the number of k-dimensional paraboloid faces of the hull process Φ which contain x, if x belongs to Vertices(Φ), and zero otherwise.

One of the main features of our approach is that $\xi_k^{(\infty)}$, $k \in \{0, 1, \ldots, d-1\}$, are indeed scaling limits of appropriately re-scaled k-face functionals, as seen in Lemmas 4.6 and 4.7 of Sect. 4.

Define the following second order correlation functions for $\xi^{(\infty)}(x, \mathcal{P}) := \xi_k^{(\infty)}(x, \mathcal{P})$ (cf. (7.2), (7.3) of [8]).

Definition 2.3 For all $w_1, w_2 \in \mathbb{R}^d$, put

$$\begin{aligned} \zeta_{\xi^{(\infty)}}(w_1, w_2) &:= \zeta_{\xi^{(\infty)}}(w_1, w_2, \mathcal{P}) \\ &:= \mathbb{E}\xi^{(\infty)}(w_1, \mathcal{P} \cup \{w_2\})\xi^{(\infty)}(w_2, \mathcal{P} \cup \{w_1\}) \\ &\quad -\mathbb{E}\xi^{(\infty)}(w_1, \mathcal{P})\mathbb{E}\xi^{(\infty)}(w_2, \mathcal{P}). \end{aligned} \tag{2.2}$$

Note that

$$\sigma^2(\xi^{(\infty)}) := \int_0^\infty \mathbb{E}\xi^{(\infty)}((\mathbf{0}, h), \mathcal{P})^2 dh + \int_0^\infty \int_{\mathbb{R}^{d-1}} \int_0^\infty \zeta_{\xi^{(\infty)}}((\mathbf{0}, h), (v', h')) dh' dv' dh \tag{2.3}$$

is finite and positive by Theorems 7.1 and 7.3 in [8].

Theorem 1.1 is a special case of the following general result expressing the asymptotic behavior of the empirical k-face measures in terms of parabolic growth processes. Let $\mathcal{C}(K)$ be the class of continuous functions on K and let $\langle g, \mu_\lambda^\xi \rangle$ denote the integral of g with respect to μ_λ^ξ.

Theorem 2.1 *For all $k \in \{0, 1, \ldots, d-1\}$, $K \in \mathcal{K}_+^3$, and $g \in \mathcal{C}(K)$ we have*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \mathbb{E}[\langle g, \mu_\lambda^{\xi_k} \rangle] = \int_0^\infty \mathbb{E}\xi_k^{(\infty)}((\mathbf{0}, h), \mathcal{P}) dh \int_{\partial K} g(z)\kappa(z)^{1/(d+1)} dz \tag{2.4}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \mathrm{Var}[\langle g, \mu_\lambda^{\xi_k} \rangle] = \sigma^2(\xi_k^{(\infty)}) \int_{\partial K} g(z)^2 \kappa(z)^{1/(d+1)} dz. \tag{2.5}$$

Remarks (i) *Related work*. Up to now, (2.5) has been known only for bodies of constant curvature, i.e., only for $K = r\mathbb{B}^d$, $d \geq 2$, $r > 0$; see Theorem 7.3 of [8].

(ii) *The constants*. The convergence (1.2) is implied by (2.5) with $F_{k,d} = \sigma^2(\xi_k^{(\infty)})$. Indeed, applying (2.1) to $g \equiv 1$, we have

$$\langle 1, \mu_\lambda^{\xi_k} \rangle = \sum_{x \in \mathcal{P}_\lambda \cap K} \xi_k(x, \mathcal{P}_\lambda \cap K) = f_k(K_\lambda).$$

(iii) *Extensions*. As in [14] and [6], we expect that our main results hold for $K \in \mathcal{K}_+^2$ and we comment on this at the end of Sect. 5.2. Following the methods of Sect. 6, we may obtain the counterpart of Theorem 2.1 for binomial input.

3 Affine and scaling transformations

Fix $K \in \mathcal{K}_+^3$. For each $z \in \partial K$, we first consider an affine transformation $\mathcal{A}_z$ of K, one under which the scores ξ_k are invariant, but under which the principal curvatures of $\mathcal{A}_z(K)$ at z coincide, that is to say $\mathcal{A}_z(K)$ is 'umbilic' at z. This property allows us to readily approximate the functionals ξ_k on Poisson points in $\mathcal{A}_z(K)$ by the corresponding functionals on Poisson points in the 'osculating ball' at z, defined below. The key idea of replacing the mother body K with an osculating ball has been used by Rényi and Sulanke [17], Bárány [1], and Böröczky et al. [6], among others.

We in turn transform $\mathcal{A}_z(K)$ to a subset of $\mathbb{R}^{d-1} \times \mathbb{R}$ via scaling transforms $\mathcal{T}^{\lambda,z}$, $\lambda \geq 1$. These transforms yield re-scaled k-face functionals $\xi^{\lambda,z}$ on the Poisson points $\mathcal{T}^{\lambda,z}(\mathcal{P}_\lambda \cap \mathcal{A}_z(K))$, ones which are well approximated by re-scaled k-face functionals on the image under $\mathcal{T}^{\lambda,z}$ of Poisson points in the osculating ball at z. In the large λ limit the latter in turn converge to the scaling limit functionals $\xi^{(\infty)}$ given in Definition 2.2.

In this way the expectation and variance asymptotics for k-face functionals on Poisson points in K are obtained by averaging, with respect to all $z \in \partial K$, the respective asymptotics for the re-scaled k-face functionals on Poisson points in osculating balls at z. The limit theory of the latter is established in [8,19] and we shall draw upon it in our approach.

3.1 Affine transformations $\mathcal{A}_z$, $z \in K$

Fix $K \in \mathcal{K}^3_+$. Let $\mathcal{M}(K)$ be the medial axis of K, i.e. the set of points x in the interior of K such that there exists a maximal ball centered at x and included in K. In other words, $\mathcal{M}(K)$ is the set of interior points having more than one closest point on ∂K. $\mathcal{M}(K)$ has Lebesgue measure zero and we parameterize points $x \in K \setminus \mathcal{M}(K)$ by $x := (z, t)$, where $z \in \partial K$ is the unique boundary point closest to x and where $t \in [0, \infty)$ is the distance between x and z.

Denote by $C_{z,1}, \ldots, C_{z,d-1}$ the principal curvatures of ∂K at z, i.e. the eigenvalues of the Weingarten operator at z. Let $\kappa(z) := \prod_{i=1}^{d-1} C_{z,i}$ be the Gaussian curvature at z, so that the *Gaussian curvature radius* r_z satisfies $\kappa(z) = r_z^{-(d-1)}$.

For $z \in \partial K$, consider the affine transformation $\mathcal{A}_z$ which preserves z, the Lebesgue measure, the unit inner normal to z, and which transforms the Weingarten operator at z into $r_z^{-1} I_{d-1}$ where I_{d-1} is the identity matrix of $\mathbb{R}^{d-1}$. Under the action of $\mathcal{A}_z$, the number of k-faces of the random convex hull inside the mother body K is preserved. Additionally, $\xi_k, k \in \{0, 1, \ldots, d-1\}$ is stable under the action of $\mathcal{A}_z$, namely

$$\xi_k(x, \mathcal{P}_\lambda \cap K) = \xi_k(\mathcal{A}_z(x), \mathcal{A}_z(\mathcal{P}_\lambda \cap K)). \tag{3.1}$$

Indeed, $\mathcal{A}_z$ sends any k-face of K_λ to a k-face of $\mathcal{A}_z(K_\lambda)$. This follows since affine transformations preserve convexity and convex hulls. A k-face F_k of K_λ is a.s. the convex hull of $(k+1)$ points from $\mathcal{P}_\lambda$, so it is sent to the convex hull of the images by $\mathcal{A}_z$. Moreover, any support hyperplane H such that $H \cap K_\lambda = F_k$ is sent to a support hyperplane of the image of K_λ such that its intersection with it is the image of the face F_k. So the image of F_k is also a k-face of the image of K_λ.

Put $K_z := \mathcal{A}_z(K)$. By construction the principal curvatures at z all equal r_z^{-1}. We recall that $\mathcal{A}_z$ preserves the distribution of $\mathcal{P}_\lambda$ so in the sequel, we shall make a small abuse of notation by identifying $\mathcal{P}_\lambda$ and K_λ with $\mathcal{A}_z(\mathcal{P}_\lambda)$ and $\mathcal{A}_z(K_\lambda)$, respectively. Let $B_r(x)$ denote the Euclidean ball of radius r centered at x. Define the *osculating ball* at $z \in \partial K$ to be the ball whose center $z_0 := z_0(z)$ is at distance r_z from z along the inner normal to z. Lemma 4.4 shows that the boundary of the osculating ball $B_{r_z}(z_0)$ is not far from ∂K_z, justifying the terminology.

Given $z \in \partial K$, define $f : \mathbb{S}^{d-1} \mapsto \mathbb{R}^+$ to be the function such that for all $u \in \mathbb{S}^{d-1}$, $(z_0 + f(u)u)$ is the point of the half line $(z_0 + \mathbb{R}^+ u)$ contained in ∂K_z and furthest from z_0. Thus ∂K_z is given by $(f(u), u)$, $u \in \mathbb{S}^{d-1}$. Given $z \in \partial K$ we let the inner unit normal be $k_z := (z_0 - z)/|z - z_0|$. Here and elsewhere we let $|w|$ denote the Euclidean norm of w. For each fixed $z \in \partial K$, we parameterize points w in $\mathbb{R}^{d-1} \times \mathbb{R}$ by (r, u) where $r := |w - z_0|$ and where $\in \mathbb{S}^{d-1}$. Henceforth, points (r, u) are with reference to z. For $z = (r_z, u_z) \in \partial K$ (where $u_z = -k_z$), let $T_z \sim \mathbb{R}^{d-1}$ denote the tangent space to $\mathbb{S}^{d-1}$ at u_z. The exponential map $\exp_{d-1} : T_z \to \mathbb{S}^{d-1}$ maps a vector v of the tangent space to the point $u \in \mathbb{S}^{d-1}$ such that u lies at the end of the geodesic of length $|v|$ starting at z and having direction v. We let the origin of the tangent space be at u_z.

3.2 Re-scaled k-face functionals $\mathcal{T}^{\lambda,z}$, $z \in \partial K$, $\lambda \geq 1$

Having transformed K to K_z, we now re-scale K_z for all $\lambda \geq 1$ with scaling transform denoted $\mathcal{T}^{\lambda,z}$. Our choice of $\mathcal{T}^{\lambda,z}$ is motivated by the following desiderata. First, consider the epigraph of $s_\lambda : \mathbb{S}^{d-1} \mapsto \mathbb{R}$ defined by

$$s_\lambda(u, \mathcal{P}_\lambda) = r_z - h_{K_\lambda}(u), \ u \in \mathbb{S}^{d-1},$$

where we recall that r_z is the Gaussian curvature radius at z and $h_{K_\lambda}(u) := \sup\{\langle x, u\rangle, \ x \in K_\lambda\}$ denotes the support function of K_λ. Noting that $h_{K_\lambda}(u) = \sup_{x \in \mathcal{P}_\lambda} h_x(u)$ for $u \in \mathbb{S}^{d-1}$, it follows that the considered epigraph is the union of epigraphs, which, locally near the vertices of K_λ, are of parabolic structure. Thus any scaling transform should preserve this structure, as should the scaling limit. Second, a subset of K_z close to z and having a unit volume scaling image should host on average $\Theta(1)$ points of the re-scaled points, that is to say the intensity density of the re-scaled points should be of order $\Theta(1)$. As in Section 2 of [8], it follows that the transform $\mathcal{T}^{\lambda,z}$ should re-scale K_z in the $(d-1)$ tangential directions with factor $\lambda^{1/(d+1)}$ and in the radial direction with factor $\lambda^{2/(d+1)}$. It is easily checked that the following choice of $\mathcal{T}^{\lambda,z}$ meet these criteria; cf. Lemma 3.1 below. Throughout we put

$$\beta := \frac{1}{d+1}.$$

Define for all $z \in \partial K$ and $\lambda \geq 1$ the finite-size scaling transformation $\mathcal{T}^{\lambda,z} : \mathbb{R}^+ \times \mathbb{S}^{d-1} \to \mathbb{R}^{d-1} \times \mathbb{R}$ by

$$\mathcal{T}^{\lambda,z}\left((r, u)\right) := \left(\left(r_z^d \lambda\right)^\beta \exp_{d-1}^{-1}(u), \left(r_z^d \lambda\right)^{2\beta}\left(1 - \frac{r}{r_z}\right)\right) := \left(v', h'\right) := w'. \tag{3.2}$$

Here $\exp_{d-1}^{-1}(\cdot)$ is the inverse exponential map, which is well defined on $\mathbb{S}^{d-1} \setminus \{-u_z\}$ and which takes values in the ball of radius π and centered at the origin of the tangent space T_z. We shall write $v' := (r_z^d \lambda)^\beta \exp_{d-1}^{-1} u := (r_z^d \lambda)^\beta v$, where $v \in \mathbb{R}^{d-1}$.

We put

$$\mathcal{T}^{\lambda,z}(K_z) := K^{\lambda,z}; \ \mathcal{T}^{\lambda,z}(B_{r_z}(z_0)) := B^{\lambda,z};$$
$$\mathcal{T}^{\lambda,z}(\mathcal{P}_\lambda \cap \mathcal{K}_z) := \mathcal{P}^{\lambda,z}; \quad \mathcal{T}^{\lambda,z}(\mathcal{P}_\lambda \cap B_{r_z}(z_0)) := \mathcal{P}_{r_z}^{\lambda,z}.$$

We also have the a.e. equality $B^{\lambda,z} = (r_z^d \lambda)^\beta \mathbb{B}_{d-1}(\pi) \times [0, (r_z^d \lambda)^{2\beta})$, where $\mathbb{B}_{d-1}(\pi)$ is the closure of the injectivity region of $\exp_{d-1}$.

We next use the scaling transformations $\mathcal{T}^{\lambda,z}$ on $\mathcal{A}_z(K)$ to define re-scaled k-face functionals $\xi^{\lambda,z}$ on re-scaled point sets $\mathcal{T}^{\lambda,z}(\mathcal{P}_\lambda \cap \mathcal{K}_z)$; in the sequel we show that these re-scaled functionals converge to the scaling limit functional $\xi^{(\infty)}$ given in Definition 2.2. In the special case that K is a ball, we remark that $\mathcal{A}_z(\mathcal{K}) = K$ for all $z \in \partial K$ and that $\mathcal{T}^{\lambda,z}$ coincide for all $z \in \partial K$, putting us in the set-up of [8].

3.3 Re-scaled k-face functionals $\xi^{\lambda,z}$, $z \in \partial K$, $\lambda \geq 1$

Fix $K \in \mathcal{K}_+^3$, $\lambda \in [1, \infty)$ and $z \in \partial K$. *Let* $\xi_k, k \in \{0, 1, \dots, d-1\}$, *be a generic k -face functional,* as in Definition 2.1. The inverse transformation $[\mathcal{T}^{\lambda,z}]^{-1}$ defines generic re-scaled k-face functionals $\xi^{\lambda,z}(w', \mathcal{X})$ defined for $w' \in K^{\lambda,z}$ and $\mathcal{X} \subset \mathbb{R}^d$ by

$$\xi^{\lambda,z}(w', \mathcal{X}) := \xi_k^{\lambda,z}(w', \mathcal{X}) := \xi_k([\mathcal{T}^{\lambda,z}]^{-1}(w'), [\mathcal{T}^{\lambda,z}]^{-1}(\mathcal{X} \cap K^{\lambda,z})). \quad (3.3)$$

It follows for all $k \in \{0, 1, \dots, d-1\}$, $z \in \partial K$, $\lambda \in [1, \infty)$, and $x \in K_z$ that $\xi_k(x, \mathcal{P}_\lambda \cap K_z) := \xi_k^{\lambda,z}(\mathcal{T}^{\lambda,z}(x), \mathcal{P}^{\lambda,z})$.

We shall establish properties of the re-scaled k-face functionals in the next section. For now, we record the distributional limit of the re-scaled point processes $\mathcal{P}_{r_z}^{\lambda,z}$ as $\lambda \to \infty$.

Lemma 3.1 *Fix $z \in \partial K$. As $\lambda \to \infty$, there is a rate one homogeneous Poisson point process $\tilde{\mathcal{P}}$ such that $\mathcal{P}_{r_z}^{\lambda,z} \xrightarrow{\mathcal{D}} \tilde{\mathcal{P}}$ in the sense of total variation convergence on compact sets.*

Proof This proof is a consequence of the discussion around (2.14) of [8], but for the sake of completeness we include the details. We find the image by $\mathcal{T}^{\lambda,z}$ of the measure on $B_{r_z}(z_0)$ given by $\lambda r^{d-1} dr d\sigma_{d-1}(u)$. Under $\mathcal{T}^{\lambda,z}$ we have $h' := (r_z^d \lambda)^{2\beta}(1 - \frac{r}{r_z})$, whence $r = r_z(1 - (r_z^d \lambda)^{-2\beta} h')$. Likewise we have $v' := (r_z^d \lambda)^{\beta} v$, whence $v = (r_z^d \lambda)^{-\beta} v'$. Under $\mathcal{T}^{\lambda,z}$, the measure $r^{d-1} dr$ becomes

$$r^{d-1} dr = (r_z(1 - (r_z^d \lambda)^{-2\beta}))^{d-1} r_z^{1-2\beta d} \lambda^{-2\beta} dh'$$

and $d\sigma_{d-1}(u)$ transforms to

$$d\sigma_{d-1}(u) = \frac{\sin^{d-2}((r_z^d \lambda)^{-\beta} |v'|)}{|(r_z^d \lambda)^{-\beta} v'|^{d-2}} (r_z^d \lambda)^{-1+2\beta} dv'$$

as in (2.17) of [8]. Therefore the product measure $\lambda r^{d-1} dr d\sigma_{d-1}(u)$ transforms to

$$(1 - (r_z^d \lambda)^{-2\beta} h')^{d-1} \frac{\sin^{d-2}(\lambda^{-\beta} |v'|)}{|\lambda^{-\beta} v'|^{d-2}} dh' dv'. \quad (3.4)$$

The total variation distance between Poisson measures is upper bounded by a multiple of the L^1 distance between their densities (Theorem 3.2.2 in [12]) and since $(1 - (r_z^d \lambda)^{-2\beta})^{(d-1)} \to 1$ as $\lambda \to \infty$, the result follows. □

4 Properties of the re-scaled k-face functional $\xi^{\lambda,z}$

Given $k \in \{0, 1, \dots, d-1\}$, $\lambda \geq 1$, $z \in \partial K$, we shall often write ξ for $\xi_k^{\lambda,z}$ as at (3.3).

4.1 Localization of $\xi^{\lambda,z}$

Fix $K \in \mathcal{K}^3_+$. We appeal to results of Reitzner [14] to show that the re-scaled functionals $\xi^{\lambda,z}$ 'localize', that is they are with high probability determined by 'nearby' point configurations.

For all $s > 0$ consider the inner parallel set of ∂K, namely

$$K(s) := \{x \in K \ : \ \delta^H(x, \partial K) \leq s\}, \tag{4.1}$$

with δ^H being the Hausdorff distance. Put

$$\epsilon_\lambda := (\frac{12d \log \lambda}{d_3 \lambda})^\beta, \tag{4.2}$$

where $d_3 := d_3(K)$ is as in Lemma 5 of Reitzner [14]. We begin with two localization properties of the score ξ.

Lemma 4.1 *Fix* $K \in \mathcal{K}^3_+$ *and* $k \in \{0, 1, \ldots, d-1\}$. (a) *With probability at least* $1 - O(\lambda^{-4d})$*, for all* $z \in \partial K$, $\rho \geq 1$, *we have*

$$\xi_k(x, \mathcal{P}_\lambda \cap K_z) = \begin{cases} \xi_k(x, \mathcal{P}_\lambda \cap K_z(\rho\epsilon^2_\lambda)) & \text{if } x \in K_z(\epsilon^2_\lambda) \\ 0 & \text{if } x \in K_z \setminus K_z(\epsilon^2_\lambda). \end{cases} \tag{4.3}$$

(b) *There is a constant* D_1 *such that for all* $z \in \partial K$ *and* $x \in K_z(\epsilon^2_\lambda)$ *we have*

$$P[\xi_k(x, \mathcal{P}_\lambda \cap K_z) \neq \xi_k(x, \mathcal{P}_\lambda \cap K_z \cap B_{D_1\epsilon_\lambda}(x))] = O(\lambda^{-4d}).$$

Proof Throughout we shorthand ξ_k by ξ. We prove part (a) with $\rho = 1$. The proof for $\rho > 1$ is identical. Let X_i, $i \geq 1$, be i.i.d. uniform on K_z. For every integer l, let A_l be the event that the boundary of $\text{co}(X_1, \ldots, X_l)$ is contained in $K_z(\epsilon^2_l)$.

Following nearly verbatim the discussion on p. 492 of [14], we note that $P[A^c_l]$ equals the probability that at least one facet of $\text{co}(X_1, \ldots, X_l)$ contains a point distant at least ϵ^2_l from the boundary of K_z, i.e., this is the probability that the hyperplane which is the affine hull of this facet cuts off from K_z a cap of height ϵ^2_l which contains no point from $X_1, \ldots, X_l$. By Lemma 5 of [14], the volume of this cap is bounded by $d_3\epsilon^{d+1}_l = 12d \log l / l$.

Thus when l is large enough so that $(l-d)/l > 1/2$ (ie. $l > 2d$) and $(12d \log l)/l < 1$, and using $\log(1-x) < -x$, $0 < x < 1$, we get

$$\begin{aligned} P[A^c_l] \leq \binom{l}{d} \left(1 - \frac{12d \log l}{l}\right)^{l-d} &< l^d \frac{1}{d!} \exp\left((l-d)(-\frac{12d \log l}{l})\right) \\ &\leq \frac{l^d}{d!} l^{-6d} = \frac{l^{-5d}}{d!}. \end{aligned} \tag{4.4}$$

Let A_λ be the event that the boundary of $\mathrm{co}(\mathcal{P}_\lambda \cap K)$ is contained in $K_z(\epsilon_\lambda^2)$. Letting $N(\lambda)$ be a Poisson random variable with parameter λ we compute

$$P[A_\lambda^c] = \sum_{l=0}^{\infty} P[A_l^c, N(\lambda) = l] < \sum_{|l-\lambda| \le \lambda^{3/4}} P[A_l^c] + P[|N(\lambda) - \lambda| \ge \lambda^{3/4}].$$

The last term decays exponentially with λ and so exhibits growth $O(\lambda^{-4d})$. By (4.4), the first term has the same growth bounds since

$$\sum_{|l-\lambda| \le \lambda^{3/4}} P[A_l^c] \le 2\lambda^{3/4} \max_{|\lambda - l| \le \lambda^{3/4}} P[A_l^c] \le 2\lambda^{3/4} \frac{1}{d!} (\lambda - \lambda^{3/4})^{-5d} = O(\lambda^{-4d}),$$

concluding the proof of (a).

We prove assertion (b). By part (a), it suffices to show there is $\rho_0 \ge 1$ such that for $x \in K_z(\epsilon_\lambda^2)$

$$P[\xi(x, \mathcal{P}_\lambda \cap K_z(\rho_0 \epsilon_\lambda^2)) \ne \xi(x, \mathcal{P}_\lambda \cap K_z(\rho_0 \epsilon_\lambda^2) \cap B_{D_1 \epsilon_\lambda}(x))] = O(\lambda^{-4d}).$$

We consider the localization results described on pages 499–502 of [14] and in the Appendix of [14]. Using the set-up of Lemma 6 of [14], we choose $m := m(\lambda) := \lfloor (d_6 \lambda / (4d + 1) \log \lambda)^{(d-1)\beta} \rfloor$ points $y_1, \dots, y_m$ on ∂K_z (here $d_6 := d_6(K)$ is the constant of [14]) such that the Voronoi cells $C_{\mathrm{Vor}}(y_j)$, $1 \le j \le m$, partition K_z, and such that the diameter of $C_{\mathrm{Vor}}(y_j) \cap \partial K_z$ is $O(\epsilon_\lambda)$. Moreover, because all y_j are on ∂K_z, any bisecting hyperplane between two y_j makes an angle with ∂K_z which is bounded from below. Consequently, since the 'width' of $K_z(\epsilon_\lambda^2)$ is $O(\epsilon_\lambda^2)$, it follows that the diameter of the truncated cells $C_{\mathrm{Vor}}(y_j) \cap K_z(\epsilon_\lambda^2)$ is also $O(\epsilon_\lambda)$. Choose ρ_0 large enough so that $K_z(\rho_0 \epsilon_\lambda^2)$ contains the caps C_j, $1 \le j \le m$, given near the end of p. 498 of [14].

For all $1 \le j \le m$, let

$$S_j := \{k \in \{1, 2, \dots, m\} : \ C_{\mathrm{Vor}}(y_k) \cap C(y_j, d_{10} m^{-2\beta}) \ne \emptyset\}$$

where $C(y, h)$ denotes a cap at y of height h, and where $d_{10} := d_{10}(K)$ denotes the constant in [14]. Pages 498–500 of [14] show the existence of a set A^m such that $P[A^m] \ge 1 - c_{16} \lambda^{-4d}$, $c_{16} := c_{16}(A)$, and on A^m the score $\xi(x, \mathcal{P}_\lambda \cap K_z(\rho_0 \epsilon_\lambda^2))$ at $x \in K_z(\epsilon_\lambda^2) \cap C_{\mathrm{Vor}}(y_j)$ is determined by the Poisson points belonging to

$$U_j := U_j(x) := \bigcup_{k \in S_j} C_{\mathrm{Vor}}(y_k) \cap K_z(\epsilon_\lambda^2), \tag{4.5}$$

where $j := j(x) \in \{1, \dots, m\}$ is such that $C_{\mathrm{Vor}}(y_j)$ contains x. (Actually [14] shows this for the score $\xi(x, \mathcal{P}_\lambda \cap K_z)$ and not for $\xi(x, \mathcal{P}_\lambda \cap K_z(\rho_0 \epsilon_\lambda^2))$, but the proof is the same, since ρ_0 is chosen so that $K_z(\rho_0 \epsilon_\lambda^2)$ contains the caps C_j, $1 \le j \le m$.) By Lemma 7 of [14], the cardinality of S_j is at most $d_8 (d_{10}^{1/2} m^{-\beta} m^{\beta} + 1)^{d+1} = O(1)$, uniformly in $1 \le j \le m$. This implies that on A^m, the score $\xi(x, \mathcal{P}_\lambda \cap K_z(\rho\epsilon_\lambda^2))$ at $x \in K_z(\epsilon_\lambda^2) \cap C_{\text{Vor}}(y_j)$ is determined by the Poisson points in U_j, whose diameter is bounded by a constant multiple of the diameter of the truncated cells $C_{\text{Vor}}(y_k) \cap K_z(\rho\epsilon_\lambda^2)$, $k \in S_j$, and is thus determined by points distant at most $D_1\epsilon_\lambda$ from x, D_1 a constant. Since $P[A_m^c] \le c_{16}\lambda^{-4d}$, this proves assertion (b). □

The next lemma shows localization properties of $\xi^{\lambda,z}$. We first require more terminology.

Definition 4.1 For all $z \in \partial K$,we put

$$S^{\lambda,z} := \mathcal{T}^{\lambda,z}(K_z(\epsilon_\lambda^2) \cap B_{2D_1\epsilon_\lambda}(z)).$$

Note that if $w' = (v', h') \in S^{\lambda,z}$, then $|v'| \le D_2(\log\lambda)^\beta$ for some D_2 not depending on z (here we use $\sup_{z\in\partial K} r_z \le C$). Also, define D_3 by the relation $2[\sup_{z\in\partial K} r_z^{d\beta}]D_1\lambda^\beta\epsilon_\lambda = D_3(\log\lambda)^\beta$. For all $L > 0$ and $v \in \mathbb{R}^{d-1}$, denote by $\mathcal{C}_L(v)$ the cylinder $\{(v', h) \in \mathbb{R}^{d-1} \times \mathbb{R} : |v' - v| \le L\}$. Due to the non-linearity of $\mathcal{T}^{\lambda,z}$, localization properties for ξ do not in general imply localization properties for $\xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z})$. However, the next lemma says that if the inverse image of w' is close to z, then $\xi(w', \mathcal{P}^{\lambda,z})$ suitably localizes.

Lemma 4.2 *Fix $K \in \mathcal{K}_+^3$ and $k \in \{0, 1, \dots, d-1\}$. Uniformly in $z \in \partial K$ and $w' := (v', h') \in S^{\lambda,z}$ we have*

$$P[\xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z}) \ne \xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z} \cap \mathcal{C}_{D_3(\log\lambda)^\beta}(v'))] = O(\lambda^{-4d}).$$

Remarks When K is the unit ball we show in [8] that the scores ξ localize in the following stronger sense: for all $w' := (v', h') \in \mathcal{K}^{\lambda,z}$, there is an a.s. finite random variable $R := R(w', \mathcal{P}^{\lambda,z})$ such that

$$\xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z}) = \xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z} \cap \mathcal{C}_r(v')) \tag{4.6}$$

for all $r \ge R$, with $\sup_\lambda P[R > t] \to 0$ as $t \to \infty$. We are unable to show this latter property for arbitrary smooth K.

Proof Fix the reference boundary point $z \in \partial K$ and write $\xi^{\lambda,z}$ for $\xi_k^{\lambda,z}$. Let ρ_0 be as in the proof of Lemma 4.1(b). For any $A \subset \mathbb{R}^+ \times \mathbb{R}^{d-1}$, we let $\mathcal{T}^{\lambda,z}(A) := A^{\lambda,z}$. In view of Lemma 4.1(b), it suffices to show for $w' := (v', h') \in S^{\lambda,z}$ that

$$P[\xi^{\lambda,z}(w', (\mathcal{P}_\lambda \cap K_z(\rho_0\epsilon_\lambda^2))^{\lambda,z}) \ne \xi^{\lambda,z}(w', (\mathcal{P}_\lambda \cap K_z(\rho_0\epsilon_\lambda^2))^{\lambda,z} \cap \mathcal{C}_{D_3(\log\lambda)^\beta}(v'))] \\ = O(\lambda^{-4d}).$$

Given w', find $j := j(w')$ such that $C_{\text{Vor}}(y_j)$ contains $[\mathcal{T}^{\lambda,z}]^{-1}(w') := x$. Recall the definition of $U_j := U_j(x)$ at (4.5) and recall that the proof of Lemma 4.1 shows that $\text{diam}(U_j) \le D_1\epsilon_\lambda$. By the C^3 assumption, if λ is large then for all $z \in \partial K$ the projection of U_j onto the osculating sphere at z has a diameter comparable to that of U_j, i.e., is generously bounded by $2D_1\epsilon(\lambda)$. Thus the spatial diameter of $\mathcal{T}^{\lambda,z}(U_j)$

is bounded by $2[\sup_{z\in\partial K} r_z^{d\beta}]\lambda^\beta D_1\epsilon_\lambda = D_3(\log\lambda)^\beta$, by definition of D_3. In other words

$$\mathcal{T}^{\lambda,z}(U_j) \subset \mathcal{C}_{D_3(\log\lambda)^\beta}(v'). \tag{4.7}$$

However, as seen in the proof of Lemma 4.1, with probability at least $1 - c_{16}\lambda^{-4d}$, the score $\xi^{\lambda,z}(w', (\mathcal{P}_\lambda \cap K_z(\rho_0\epsilon_\lambda^2))^{\lambda,z})$ is determined by the points $(\mathcal{P}_\lambda \cap K_z(\rho_0\epsilon_\lambda^2))^{\lambda,z}$ in $\mathcal{T}^{\lambda,z}(U_j)$. In view of (4.7), the proof is complete. □

4.2 Moment bounds for $\xi^{\lambda,z}$

We use the localization results to derive moment bounds for the re-scaled k-face functionals $\xi^{\lambda,z}$. Given a random variable W and $p > 0$, we let $||W||_p := (\mathbb{E}|W|^p)^{1/p}$.

Lemma 4.3 *Fix $K \in \mathcal{K}_+^3$ and $k \in \{0, 1, \dots, d-1\}$. For all $p \in [1, 4]$ there are constants $M(p) := M(p, k) \in (0, \infty)$ such that*

$$\sup_{z\in\partial K}\sup_{\lambda\geq 1}\sup_{w'\in B^{\lambda,z}} ||\xi_k^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z})||_p \leq M(p) \tag{4.8}$$

and

$$\sup_{z\in\partial K}\sup_{\lambda\geq 1}\sup_{w'\in S^{\lambda,z}} ||\xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z})||_p \leq M(p)(\log\lambda)^k. \tag{4.9}$$

Proof The bound (4.8) follows as in Lemma 7.1 of [8]. To prove (4.9), we argue as follows. Given $z \in \partial K$ and $w' \in S^{\lambda,z}$, we let

$$E := E_z(w') := \left\{\xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z}) = \xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z} \cap \mathcal{C}_{D_3(\log\lambda)^\beta}(v') \cap (K_z(\epsilon_\lambda^2))^{\lambda,z})\right\}.$$

By Lemmas 4.1(a) and 4.2 we have $P[E^c] = O(\lambda^{-4d})$.

Let $N(s)$ be a Poisson random variable with parameter s. The cardinality of the point set

$$\mathcal{P}^{\lambda,z} \cap \mathcal{C}_{D_3(\log\lambda)^\beta}(v') \cap (K_z(\epsilon_\lambda^2))^{\lambda,z},$$

is stochastically bounded by $N(C(\log\lambda)^{\beta(d-1)} \cdot (\log\lambda)^{2\beta}) = N(C\log\lambda)$, where C is a generic constant whose value may change from line to line. On the event E the number of k-faces containing w' is generously bounded by $\binom{N(C\log\lambda)}{k} \leq (N(C\log\lambda))^k$.

We now compute for $p \in [1, 4]$:

$$||\xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z})||_p \leq ||\xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z})\mathbf{1}(E)||_p + ||\xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z})\mathbf{1}(E^c)||_p.$$

The first term is bounded by $(k+1)^{-1}||N^k(C\log\lambda)||_p \leq M(p)(\log\lambda)^k$.

The second term is bounded by

$$\frac{1}{k+1}\left|\left|\binom{\mathrm{card}(\mathcal{P}_\lambda)}{k}\right|\right|_{pr}\lambda^{-4d/pq},\quad 1/r+1/q=1.$$

We have $||\binom{\mathrm{card}(\mathcal{P}_\lambda)}{k}||_{pr} = O(\lambda^k)$ and for $p \in [1, 4]$ we may choose q sufficiently close to 1 such that $\lambda^{-4d/pq} = O(\lambda^{-k})$. This gives (4.9).

Remarks (i) Straightforward modifications of the proof of Lemma 4.1 show that the $O(\lambda^{-4d})$ bounds of that lemma may be replaced by $O(\lambda^{-md})$ bounds, m an arbitrary integer, provided that ϵ_λ given at (4.2) is increased by a scalar multiple of m. In this way one could show that Lemma 4.3 holds for moments of any order $p > 0$. Since we do not require more than fourth moments for $\xi^{\lambda,z}$, we do not strive for this generality.

(ii) We do not claim that the bounds of Lemma 4.3 are optimal. By McMullen's bound [10], the k face functional on an n point set is bounded by $Cn^{d/2}$ and using this bound for $k > d/2$ shows that the $(\log\lambda)^k$ term in (4.9) can be improved to $(\log\lambda)^{d/2}$. The $\log\lambda$ factors could possibly be dispensed with altogether, as mentioned in the remark at the end of Sect. 5.2.

4.3 Comparison of scores for points in a ball and on K_z

The k-face functional of Definition 2.1 on Poisson input on the ball is well understood [8]. To exploit this we need to show that the re-scaled functional $\xi^{\lambda,z}$ on $\mathcal{P}^{\lambda,z}$ is well approximated by its value on $\mathcal{P}^{\lambda,z}_{r_z}$. We shall also need to show that the pair correlation function for $\xi^{\lambda,z}$ on $\mathcal{P}^{\lambda,z}$ is well approximated by the pair correlation function for $\xi^{\lambda,z}$ on $\mathcal{P}^{\lambda,z}_{r_z}$. These approximations are established in the next four lemmas.

Our first lemma records a simple geometric fact. Locally around z, the osculating ball to K_z may lie inside or outside K_z, but it is not far from ∂K_z. The next lemma shows that the distance decays like the cube of $|v'|$.

Lemma 4.4 *Fix $K \in \mathcal{K}^3_+$. For all $z \in \partial K$ and $v := (r_z^d\lambda)^{-\beta}v'$ we have*

$$r_z^{2\beta d}\lambda^{2\beta}\left|1-\frac{f(\exp_{d-1}(v))}{r_z}\right| \le D_4 r_z^{-1-\beta d}\lambda^{-\beta}|v'|^3. \tag{4.10}$$

Proof We first show (4.10) when $d = 2$. The boundary of the osculating circle at z coincides with ∂K up to at least second order, giving $f(0) = r_z$, $f'(0) = f''(0) = 0$. The Taylor expansion for f around 0 gives $|1 - \frac{f(v)}{r_z}| \le \frac{1}{6}||f'''||_\infty r_z^{-1}|v|^3$, whence the result.

Consider the case $d \ge 3$. Let $\exp_{d-1}(v) := \cos(|v|)k_z + \sin(|v|)w$, where $w := v/|v|$. It is enough to consider the section of the osculating ball and K_z with the plane generated by k_z and w. Indeed, we obtain in that plane a two-dimensional mother body with an osculating radius equal to r_z at the point z. We may apply the case $d = 2$ to deduce the required result. □

Lemma 4.5 *Fix* $K \in \mathcal{K}_+^3$ *and* $k \in \{0, 1, \ldots, d-1\}$*. Uniformly for* $z \in \partial K$ *and* $w' \in S^{\lambda,z} \cap B^{\lambda,z}$*, we have*

$$\mathbb{E}\left|\xi_k^{\lambda,z}(w', \mathcal{P}^{\lambda,z}) - \xi_k^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z})\right| = O\left(\lambda^{-\beta/2}(\log\lambda)^{k+(\beta+1)/2}\right). \quad (4.11)$$

Proof Write $\xi^{\lambda,z}$ for $\xi_k^{\lambda,z}$. For $w' \in S^{\lambda,z} \cap B^{\lambda,z}$, we put

$$\begin{aligned} E := E(w') := \{&\xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z}) = \xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z} \cap \mathcal{C}_{D_3(\log\lambda)^\beta}(w'))\\ &\cup \{\xi^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z}) = \xi^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z} \cap \mathcal{C}_{D_3(\log\lambda)^\beta}(w'))\}, \end{aligned} \quad (4.12)$$

so that $P[E^c] = O(\lambda^{-4d})$ by Lemma 4.2. Put

$$F^{\lambda,z}(w') := \xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z}) - \xi^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z}).$$

By Lemma 4.3 with $p = 2$, we have $||F^{\lambda,z}(w')||_2 \leq 2M(2)(\log\lambda)^k$, uniformly in w', λ and z.

Recall $w' := (v', h')$. For all $w' \in S^{\lambda,z} \cap B^{\lambda,z}$ put

$$\begin{aligned} R(w') := \{&(v'', h'') \in \mathbb{R}^{d-1} \times \mathbb{R} : \ |v'' - v'| \leq D_3(\log\lambda)^\beta,\\ &|h''| \leq (r_z^d\lambda)^{2\beta}|1 - r_z^{-1} f(\exp_{d-1}((r_z^d\lambda)^{-\beta}v''))|\}. \end{aligned} \quad (4.13)$$

Write

$$\mathbb{E}|F^{\lambda,z}(w')| = \mathbb{E}|(F^{\lambda,z}(w'))(\mathbf{1}(E) + \mathbf{1}(E^c))|.$$

On E we have $F^{\lambda,z}(w') = 0$, unless the realization of $\mathcal{P}^{\lambda,z}$ puts points in the set $R(w')$. By the Cauchy–Schwarz inequality and Lemma 4.3 with $p = 2$ there, we have

$$\mathbb{E}|(F^{\lambda,z}(w'))\mathbf{1}(E)| \leq 2M(2)(\log\lambda)^k (P[\mathbf{1}(\mathcal{P}^{\lambda,z} \cap R(v') \neq \emptyset)])^{1/2}. \quad (4.14)$$

The Lebesgue measure of $R(w')$ is bounded by the product of the area of its ‘base’, that is $(2D_3(\log\lambda)^\beta)^{d-1}$ and its ‘height’, which by Lemma 4.4 is at most $D_4 r_z^{-1-\beta d}\lambda^{-\beta}\left(|v'| + D_3(\log\lambda)^\beta\right)^3$.

By (3.4), the $\mathcal{P}^{\lambda,z}$ intensity measure of $R(w')$, denoted by $|R(w')|$, thus satisfies

$$|R(w')| \leq (2D_3(\log\lambda)^\beta)^{d-1} D_4 r_z^{-1-\beta d}\lambda^{-\beta}(|v'| + D_3(\log\lambda)^\beta)^3. \quad (4.15)$$

Since $1 - e^{-x} \leq x$ holds for all x it follows that

$$P[\mathbf{1}(\mathcal{P}^{\lambda,z} \cap R(v') \neq \emptyset)] = 1 - \exp(-|R(w')|) \leq |R(w')|. \quad (4.16)$$

Combining (4.14)–(4.16), and recalling that $|v'| \leq D_2(\log\lambda)^\beta$, shows that $\mathbb{E}|(F^{\lambda,z}(w'))\mathbf{1}(E)|$ is bounded by the right hand side of (4.11).

Similarly, Lemma 4.3, the bound $P[E^c] = O(\lambda^{-4d})$, and the Cauchy–Schwarz inequality give $\mathbb{E}|(F^{\lambda,z}(w'))\mathbf{1}(E^c)| = O((\log \lambda)^k \lambda^{-2d})$, which is dominated by the right hand side of (4.11). Thus (4.11) holds as claimed. □

The next two lemmas justify identifying $\xi^{(\infty)}$ as a scaling limit, as given by Definition 2.2. The first lemma is a restatement of Lemma 7.2 in [8].

Lemma 4.6 *For all* $k \in \{0, 1, \ldots, d-1\}$, $z \in \partial K$ *and* $h \geq 0$ *we have*

$$\lim_{\lambda\to\infty} |\mathbb{E}\xi_k^{\lambda,z}((\mathbf{0}, h), \mathcal{P}_{r_z}^{\lambda,z}) - \mathbb{E}\xi_k^{(\infty)}((\mathbf{0}, h), \mathcal{P})| = 0.$$

Lemma 4.7 *Fix* $K \in \mathcal{K}_+^3$. *For all* $k \in \{0, 1, \ldots, d-1\}$, $z \in \partial K$ *and* $(\mathbf{0}, h) \in K^{\lambda,z}$ *we have*

$$\lim_{\lambda\to\infty} |\mathbb{E}\xi_k^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z}) - \mathbb{E}\xi_k^{(\infty)}((\mathbf{0}, h), \mathcal{P})| = 0.$$

Proof We bound $|\mathbb{E}\xi_k^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z}) - \mathbb{E}\xi_k^{(\infty)}((\mathbf{0}, h), \mathcal{P})|$ by

$$|\mathbb{E}\xi_k^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z}) - \mathbb{E}\xi_k^{\lambda,z}((\mathbf{0}, h), \mathcal{P}_{r_z}^{\lambda,z})| + |\mathbb{E}\xi_k^{\lambda,z}((\mathbf{0}, h), \mathcal{P}_{r_z}^{\lambda,z}) - \mathbb{E}\xi_k^{(\infty)}((\mathbf{0}, h), \mathcal{P})|.$$

The first term goes to zero by Lemma 4.5 with $w' = (\mathbf{0}, h)$ and the second term goes to zero by Lemma 4.6. □

We next recall the definition of the pair correlation function for the score ξ as well as for its re-scaled version.

Definition 4.2 *(Pair correlation functions)* For all $x, y \in K_z$, any random point set $\Xi \subset K_z$, and any ξ we put

$$c(x, y; \Xi) := c^{\xi}(x, y; \Xi) := \mathbb{E}\xi(x, \Xi \cup y)\xi(y, \Xi \cup x) - \mathbb{E}\xi(x, \Xi)\mathbb{E}\xi(y, \Xi). \tag{4.17}$$

For all $k \in \{0, 1, \ldots, d-1\}$, $\lambda \geq 1$, $z \in \partial K$, $(\mathbf{0}, h) \in K^{\lambda,z}$, and $(v', h') \in K^{\lambda,z}$, define the re-scaled pair correlation function for $\xi_k^{\lambda,z}$ with respect to $\mathcal{P}^{\lambda,z}$ by

$$\begin{aligned}
&c^{\lambda,z}((\mathbf{0}, h), (v', h'); \mathcal{P}^{\lambda,z}) \\
&\quad := c_k^{\lambda,z}((\mathbf{0}, h), (v', h'); \mathcal{P}^{\lambda,z}) \\
&\quad = \mathbb{E}\xi_k^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z} \cup (v', h'))\xi_k^{\lambda,z}((v', h'), \mathcal{P}^{\lambda,z} \cup (\mathbf{0}, h)) \\
&\qquad - \mathbb{E}\xi_k^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z})\mathbb{E}\xi_k^{\lambda,z}((v', h'), \mathcal{P}^{\lambda,z}).
\end{aligned} \tag{4.18}$$

The next lemma shows that the pair correlation function for $\xi^{\lambda,z}$ on $\mathcal{P}^{\lambda,z}$ is well approximated by the pair correlation function for $\xi^{\lambda,z}$ on $\mathcal{P}_{r_z}^{\lambda,z}$.

Lemma 4.8 *Fix* $K \in \mathcal{K}_+^3$ *and* $k \in \{0, 1, \ldots, d-1\}$. *Uniformly for* $z \in \partial K$, $w_0' := (\mathbf{0}, h) \in S^{\lambda,z} \cap B^{\lambda,z}$ *and* $w' := (v', h') \in S^{\lambda,z} \cap B^{\lambda,z}$, *we have*

$$|c_k^{\lambda,z}(w_0', w'; \mathcal{P}^{\lambda,z}) - c_k^{\lambda,z}(w_0', w'; \mathcal{P}_{r_z}^{\lambda,z})| = O\left(\lambda^{-\beta/3}(\log \lambda)^{2k+(\beta+1)/3}\right). \tag{4.19}$$

Proof It suffices to modify the proof of Lemma 4.5. Write $c^{\lambda,z}$ and $\xi^{\lambda,z}$ for $c_k^{\lambda,z}$ and $\xi_k^{\lambda,z}$, respectively. Put $F := E(w_0{}') \cap E(w')$, where $E(w_0{}')$ and $E(w')$ are defined at (4.12). We have $P[F^c] = O(\lambda^{-4d})$ by Lemma 4.2. Write

$$\begin{aligned}
&\mathbb{E}\xi^{\lambda,z}(w_0{}', \mathcal{P}^{\lambda,z} \cup (v', h'))\xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z} \cup w_0{}') \\
&\quad -\mathbb{E}\xi^{\lambda,z}(w_0{}', \mathcal{P}_{r_z}^{\lambda,z} \cup w')\xi^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z} \cup w_0{}') \\
&= \mathbb{E}[\{\xi^{\lambda,z}(w_0{}', \mathcal{P}^{\lambda,z} \cup w')\xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z} \cup w_0{}') \\
&\quad -\xi^{\lambda,z}(w_0{}', \mathcal{P}_{r_z}^{\lambda,z} \cup w')\xi^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z} \cup w_0{}')\}\mathbf{1}(F)] \\
&\quad + \mathbb{E}[\{\xi^{\lambda,z}(w_0{}', \mathcal{P}^{\lambda,z} \cup w')\xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z} \cup w_0{}') \\
&\quad -\xi^{\lambda,z}(w_0{}', \mathcal{P}_{r_z}^{\lambda,z} \cup w')\xi^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z} \cup w_0{}')\}\mathbf{1}(F^c)] \\
&:= I_1 + I_2.
\end{aligned} \tag{4.20}$$

The random variable in the expectation I_1 vanishes, except on the event

$$H(w_0{}', w') := \{\mathcal{P}^{\lambda,z} \cap R(w') \neq \emptyset\} \cup \{\mathcal{P}^{\lambda,z} \cap R(w_0{}') \neq \emptyset\},$$

where $R(w')$ and $R(w_0{}')$ are at (4.13). The Hölder inequality $||UVW||_1 \leq ||U||_3||V||_3||W||_3$ for random variables U, V, W and Lemma 4.3 with $p = 3$ imply that

$$I_1 \leq 2(M(3))^2(\log\lambda)^{2k}(P[H(w_0{}', w')])^{1/3},$$

that is to say

$$I_1 = O\left((\log\lambda)^{2k}\left(r_z^{-1-\beta d}\lambda^{-\beta}(\log\lambda)^{\beta(d-1)}[(|v'| + D_3(\log\lambda)^{\beta})^3 + (D_3(\log\lambda)^{\beta})^3]\right)^{1/3}\right),$$

which for $|v'| \leq D_2(\log\lambda)^{\beta}$ satisfies the growth bounds on the right hand side of (4.19).

Now term I_2 in (4.20) is bounded by $2(M(3))^2(\log\lambda)^{2k}(P[F^c])^{1/3}$, which is of smaller order than the right hand side of (4.19). This shows that (4.20) also satisfies the growth bounds on the right hand side of (4.19).

It remains to bound

$$|\mathbb{E}\xi^{\lambda,z}(w_0{}', \mathcal{P}^{\lambda,z})\mathbb{E}\xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z}) - \mathbb{E}\xi^{\lambda,z}(w_0{}', \mathcal{P}_{r_z}^{\lambda,z})\mathbb{E}\xi^{\lambda,z}(w', \mathcal{P}_{r_z}^{\lambda,z})|. \tag{4.21}$$

Notice that two applications of the Cauchy–Schwarz inequality give that the difference (4.21) differs from

$$\begin{aligned}
&|\mathbb{E}[\xi^{\lambda,z}(w_0{}', \mathcal{P}^{\lambda,z})\mathbf{1}(F)]\mathbb{E}[\xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z})\mathbf{1}(F)] \\
&\quad -\mathbb{E}[\xi^{\lambda,z}(w_0{}', \mathcal{P}_{r_z}^{\lambda,z})\mathbf{1}(F)]\mathbb{E}[\xi(w', \mathcal{P}_{r_z}^{\lambda,z})\mathbf{1}(F)]|
\end{aligned} \tag{4.22}$$

by at most

$$4M(1)M(2)(\log\lambda)^{2k}P[F^c]^{1/2} = O((\log\lambda)^{2k}\lambda^{-2d}),$$

which is of smaller order than the right hand side of (4.19).

Now we control the difference (4.22) which we write as $|\mathbb{E}e_1\mathbb{E}e_2 - \mathbb{E}e_3\mathbb{E}e_4|$, where $e_1 := \xi^{\lambda,z}(w_0', \mathcal{P}^{\lambda,z})\mathbf{1}(F)$, $e_2 := \xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z})\mathbf{1}(F)$, $e_3 := \xi^{\lambda,z}(w_0', \mathcal{P}^{\lambda,z}_{r_z})\mathbf{1}(F)$, and $e_4 := \xi^{\lambda,z}(w', \mathcal{P}^{\lambda,z}_{r_z})\mathbf{1}(F)$. The proof of Lemma 4.5 (with E replaced by F) shows that

$$\mathbb{E}|e_1 - e_3| = O(\lambda^{-\beta/2}(\log\lambda)^{k+(\beta+1)/2})$$

and

$$\mathbb{E}|e_2 - e_4| = O(\lambda^{-\beta/2}(\log\lambda)^{k+(\beta+1)/2}).$$

Since $|\mathbb{E}e_1\mathbb{E}e_2 - \mathbb{E}e_3\mathbb{E}e_4| \le |\mathbb{E}e_1||\mathbb{E}e_2 - \mathbb{E}e_4| + |\mathbb{E}e_4||\mathbb{E}e_1 - \mathbb{E}e_3|$ it follows that (4.21) is bounded by

$$O(\lambda^{-\beta/2}(\log\lambda)^{2k+(\beta+1)/2}) + O((\log\lambda)^{2k}\lambda^{-4d/3}),$$

i.e., is bounded by the right-hand side of (4.19). □

Letting $\xi := \xi_k$ denote a generic k-face functional, our last lemma describes a decay rate for $c^{\xi}(x, y; \mathcal{P}_\lambda \cap K_z)$ which, while not optimal, is enough for proving Theorem 2.1. □

Lemma 4.9 *Fix $K \in \mathcal{K}^3_+$. For all $z \in \partial K$ and $x, y \in K_z(\epsilon_\lambda^2)$ with $|x - y| \ge 2D_1\epsilon_\lambda$, we have*

$$\lim_{\lambda\to\infty} \lambda^{1+2\beta} c^{\xi}(x, y; \mathcal{P}_\lambda \cap K_z) = 0.$$

Proof Fix $x \in K_z(\epsilon_\lambda^2)$. *To lighten the notation we abbreviate* $\mathcal{P}_\lambda \cap K_z$ by $\mathcal{P}_\lambda$ *in this proof only.* For $y \in K_z(\epsilon_\lambda^2)$, put

$$E := E(x, y) := \{\xi(x, \mathcal{P}_\lambda) = \xi(x, \mathcal{P}_\lambda \cap B_{D_1\epsilon_\lambda}(x))\} \cup \{\xi(y, \mathcal{P}_\lambda) = \xi(y, \mathcal{P}_\lambda \cap B_{D_1\epsilon_\lambda}(y))\}.$$

Lemma 4.1(b) gives

$$P[E^c] = O(\lambda^{-4d}). \tag{4.23}$$

If $|x - y| \ge 2D_1\epsilon_\lambda$, then $\xi(x, \mathcal{P}_\lambda \cup y)$ and $\xi(y, \mathcal{P}_\lambda \cup x)$ are independent on E, giving

$$\begin{aligned}\mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup y)\xi(y, \mathcal{P}_\lambda \cup x)\mathbf{1}(E)] &= \mathbb{E}[\xi(x, \mathcal{P}_\lambda)\mathbf{1}(E) \cdot \xi(y, \mathcal{P}_\lambda)\mathbf{1}(E)] \\ &= \mathbb{E}[\xi(x, \mathcal{P}_\lambda)\mathbf{1}(E)] \cdot \mathbb{E}[\xi(y, \mathcal{P}_\lambda)\mathbf{1}(E)].\end{aligned}$$

Writing $\mathbf{1}(E) = 1 - \mathbf{1}(E^c)$ gives

$$\begin{aligned}&\mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup y)\xi(y, \mathcal{P}_\lambda \cup x)\mathbf{1}(E)] \\ &\quad = \left(\mathbb{E}\xi(x, \mathcal{P}_\lambda) - \mathbb{E}[\xi(x, \mathcal{P}_\lambda)\mathbf{1}(E^c)]\right) \cdot \left(\mathbb{E}\xi(y, \mathcal{P}_\lambda) - \mathbb{E}[\xi(y, \mathcal{P}_\lambda)\mathbf{1}(E^c)]\right) \\ &\quad = \mathbb{E}\xi(x, \mathcal{P}_\lambda)\mathbb{E}\xi(y, \mathcal{P}_\lambda) + G(x, y),\end{aligned}$$

where

$$\begin{aligned}G(x, y) := &-\mathbb{E}\xi(x, \mathcal{P}_\lambda)\mathbb{E}[\xi(y, \mathcal{P}_\lambda)\mathbf{1}(E^c)] \\ &- \mathbb{E}\xi(y, \mathcal{P}_\lambda)\mathbb{E}[\xi(x, \mathcal{P}_\lambda)\mathbf{1}(E^c)] + \mathbb{E}[\xi(x, \mathcal{P}_\lambda)\mathbf{1}(E^c)] \cdot \mathbb{E}[\xi(y, \mathcal{P}_\lambda)\mathbf{1}(E^c)].\end{aligned}$$

Let $N(\lambda) := \text{card}(\mathcal{P}_\lambda \cap K_z)$. By McMullen's bounds [10] for the number of k-dimensional faces and standard moment bounds for Poisson random variables we have $||\xi(x, \mathcal{P}_\lambda)||_1 \le C||N^{d/2}(\lambda)||_1 \le C\lambda^{d/2}$ and similarly $||\xi(y, \mathcal{P}_\lambda)||_2 \le C\lambda^{d/2}$. By the Cauchy–Schwarz inequality, it follows that

$$|\mathbb{E}\xi(x, \mathcal{P}_\lambda)\mathbb{E}[\xi(y, \mathcal{P}_\lambda)\mathbf{1}(E^c)]| = O(\lambda^{d/2}\lambda^{d/2}(P[E^c])^{1/2}) = o(\lambda^{-1-2\beta}),$$

where the last estimate easily follows by (4.23). The other two terms comprising $G(x, y)$ have the same asymptotic behavior and so $G(x, y) = o(\lambda^{-1-2\beta})$.

On the other hand, $\mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup y)\xi(y, \mathcal{P}_\lambda \cup x)\mathbf{1}(E)]$ differs from $\mathbb{E}\xi(x, \mathcal{P}_\lambda \cup y))\xi(y, \mathcal{P}_\lambda \cup x)$ by $\mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup y)\xi(y, \mathcal{P}_\lambda \cup x)\mathbf{1}(E^c)]$. The Hölder inequality $||UVW||_1 \le ||U||_4||V||_4||W||_2$ shows that this term is $o(\lambda^{-1-2\beta})$.

Thus $\mathbb{E}\xi(x, \mathcal{P}_\lambda \cup y))\xi(y, \mathcal{P}_\lambda \cup x)$ and $\mathbb{E}\xi(x, \mathcal{P}_\lambda)\mathbb{E}\xi(y, \mathcal{P}_\lambda)$ differ from $\mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup y)\xi(y, \mathcal{P}_\lambda \cup x)\mathbf{1}(E)]$ by $o(\lambda^{-1-2\beta})$, concluding the proof of Lemma 4.9. □

5 Proof of Theorem 2.1

Fix $K \in \mathcal{K}^3_+$. Recall that $\mathcal{M}(K)$ denotes the medial axis of K and, for every $z \in \partial K$ the inner unit-normal vector of ∂K at z is k_z. Put $t(z) := \inf\{t > 0 : z + tk_z \in \mathcal{M}(K)\}$. In particular, for every $t < t(z)$, the ball $B_t(z+tk_z)$ is included in K so $t(z) < \infty$ because of the boundedness of K. Thus the map $\varphi : (z, t) \longmapsto (z + tk_z)$ is a diffeomorphism from $\{(z, t) : z \in \partial K, 0 < t < t(z)\}$ to $\text{Int}(K) \setminus \mathcal{M}(K)$. In particular, $z \longmapsto -k_z$ is the Gauss map and its ifferential is the shape operator or Weingarten map W_z, which we recall has eigenvalues $C_{z,1}, \cdots, C_{z,d-1}$. Consequently, the Jacobian of φ may be written as $\det(I - tW_z) = \prod_{i=1}^{d-1}(1 - tC_{z,i})$.

5.1 Proof of expectation asymptotics (2.4)

With $K \in \mathcal{K}^3_+$ fixed, fix $g \in \mathcal{C}(K)$ and let ξ and μ^ξ_λ denote a generic k face functional and k face measure, respectively. Recall that we may uniquely write $x \in K \setminus \mathcal{M}(K)$

as $x := (z, t)$, where $z \in \partial K$, and $t \in (0, t(z))$ is the distance between x and z. Write

$$\lambda^{-1+2\beta}\mathbb{E}[\langle g, \mu_\lambda^\xi \rangle] = \lambda^{2\beta} \int_K g(x)\mathbb{E}\xi(x, \mathcal{P}_\lambda \cap K)dx$$

$$= \lambda^{2\beta} \int_{z\in\partial K} \int_0^{t(z)} g((z,t))\mathbb{E}\xi((z,t), \mathcal{P}_\lambda \cap K) \cdot \Pi_{i=1}^{d-1}(1 - tC_{z,i})dtdz.$$

For each $z \in \partial K$, we apply the transformation $\mathcal{A}_z$ to K. Recalling from (3.1) that ξ is stable under $\mathcal{A}_z$, we have $\mathbb{E}\xi((z,t), \mathcal{P}_\lambda \cap K) = \mathbb{E}\xi((z,t), \mathcal{P}_\lambda \cap K_z)$, since $\mathcal{A}_z(z,t) := (z,t)$ and $\mathcal{A}_z(\mathcal{P}_\lambda \cap K) \stackrel{\mathcal{D}}{=} \mathcal{P}_\lambda \cap K_z$. It follows that

$$\lambda^{-1+2\beta}\mathbb{E}[\langle g, \mu_\lambda^\xi \rangle] = \lambda^{2\beta} \int_{z\in\partial K} \int_0^{t(z)} g((z,t))\mathbb{E}\xi((z,t), \mathcal{P}_\lambda \cap K_z) \cdot \Pi_{i=1}^{d-1}(1 - tC_{z,i})dtdz.$$

By Lemma 4.1(a), the bound (4.9) with $p = 2$, and the Cauchy–Schwarz inequality, it follows that uniformly in $x \in K_z \setminus K_z(\epsilon_\lambda^2)$ we have $\lim_{\lambda\to\infty} \lambda^{2\beta}\mathbb{E}\xi(x, \mathcal{P}_\lambda \cap K_z) = 0$. Since

$$\sup_{\lambda\geq 1} \sup_{x\in K_z\setminus K_z(\epsilon_\lambda^2)} \lambda^{2\beta}\mathbb{E}\xi(x, \mathcal{P}_\lambda \cap K_z) \leq C,$$

the bounded convergence theorem shows that we can restrict the range of integration of t to the interval $[0, \epsilon_\lambda^2]$ with error $o(1)$. This gives

$$\lambda^{-1+2\beta}\mathbb{E}[\langle g, \mu_\lambda^\xi \rangle] = \lambda^{2\beta} \int_{z\in\partial K} \int_0^{\epsilon_\lambda^2} g((z,t))\mathbb{E}\xi((z,t), \mathcal{P}_\lambda \cap K_z) \cdot \Pi_{i=1}^{d-1}(1 - tC_{z,i})dtdz + o(1). \tag{5.1}$$

Changing variables with $t = r_z(r_z^d\lambda)^{-2\beta}h$ and using $h = (r_z^d\lambda)^{2\beta}(r_z - r)/r_z = (r_z^d\lambda)^{2\beta}(t/r_z)$ gives $\xi((z,t), \mathcal{P}_\lambda \cap K_z) = \xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z})$. Letting $h(\lambda, z) := r_z^{-1+2\beta d}\lambda^{2\beta}\epsilon_\lambda^2$ we obtain

$$\lambda^{-1+2\beta}\mathbb{E}[\langle g, \mu_\lambda^\xi \rangle]$$

$$= \int_{z\in\partial K} r_z^{1-2\beta d} \int_0^{h(\lambda,z)} g((z, o(1)))\mathbb{E}\xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z}) \cdot \Pi_{i=1}^{d-1}(1 - o(1))dhdz + o(1),$$

where the first $o(1)$ denotes a quantity tending to zero as $\lambda \to \infty$, uniformly in $z \in \partial K$ and uniformly in $h \in [0, h(\lambda, z)]$.

Note that $(\mathbf{0}, h)$ belongs to $S^{\lambda,z} \cap B^{\lambda,z}$ and so we may apply Lemma 4.5 to $\xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z})$. Thus, with w' set to $(\mathbf{0}, h)$ in Lemma 4.5, we have

$$\sup_{z\in\partial K} \sup_{h\in[0,h(\lambda,z)]} h(\lambda, z) \left| \mathbb{E}\xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z}) - \mathbb{E}\xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z}_{r_z}) \right| = o(1),$$

and so we may replace $\mathbb{E}\xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z})$ by $\mathbb{E}\xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z}_{r_z})$ with error $o(1)$. We also have $r_z^{1-2\beta d} = \kappa(z)^{1/(d+1)}$. In other words,

$$\begin{aligned}
&\lambda^{-1+2\beta}\mathbb{E}[\langle g, \mu_\lambda^\xi\rangle] \\
&= \int_{z\in\partial K} \kappa(z)^{1/(d+1)} \int_0^{h(\lambda,z)} g((z, o(1))\mathbb{E}\xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z}_{r_z}) \cdot \Pi_{i=1}^{d-1}(1 - o(1))dhdz + o(1).
\end{aligned}$$

By Lemma 3.2 of [8], the integrand is dominated by an exponentially decaying function of h, uniformly in z and λ.

The continuity of g, Lemma 4.6, and the dominated convergence theorem give

$$\lim_{\lambda\to\infty} \lambda^{-1+2\beta}\mathbb{E}[\langle g, \mu_\lambda^\xi\rangle] = \int_{z\in\partial K} g(z)\kappa(z)^{1/(d+1)} \int_0^\infty \mathbb{E}\xi^{(\infty)}((\mathbf{0}, h), \mathcal{P})dhdz. \tag{5.2}$$

This gives (2.4), as desired.

5.2 Proof of variance asymptotics (2.5)

Recalling (4.17), for fixed $g \in \mathcal{C}(K)$ we have

$$\begin{aligned}
&\lambda^{-1+2\beta}\mathrm{Var}[\langle g, \mu_\lambda^\xi\rangle] \\
&\quad = \lambda^{2\beta} \int_K g(x)^2\mathbb{E}\xi^2(x, \mathcal{P}_\lambda \cap K)dx \\
&\qquad + \lambda^{1+2\beta} \int_K \int_K g(x)g(y)c(x, y; \mathcal{P}_\lambda \cap K)dydx := I_1(\lambda) + I_2(\lambda).
\end{aligned}$$

Following the proof of (2.4) until (5.2) shows that

$$\lim_{\lambda\to\infty} I_1(\lambda) = \int_{z\in\partial K} g(z)^2\kappa(z)^{1/(d+1)} \int_0^\infty \mathbb{E}(\xi^{(\infty)}((\mathbf{0}, h), \mathcal{P}))^2 dhdz. \tag{5.3}$$

Turning to $I_2(\lambda)$, write x in curvilinear coordinates (z, t) with respect to ∂K. This gives $dx = \Pi_{i=1}^{d-1}(1 - tC_{z,i})dtdz$. Apply the map $\mathcal{A}_z$, write $\mathcal{A}_z(y) = \bar{y}$ for $y \in K$, and use stability (3.1) to get

$$I_2(\lambda) = \lambda^{1+2\beta} \int_{z\in\partial K} \int_0^{t(z)} \int_{\bar{y}\in K_z} g((z,t))g(\bar{y})c((z,t),\bar{y};\mathcal{P}_\lambda \cap K_z)d\bar{y} \cdot \times \Pi_{i=1}^{d-1}(1 - tC_{z,i})dtdz. \tag{5.4}$$

Here

$$c((z,t),\bar{y};\mathcal{P}_\lambda \cap K_z) = \mathbb{E}\xi((z,t),\mathcal{P}_\lambda \cap K_z \cup \{\bar{y}\})\xi(\bar{y},\mathcal{P}_\lambda \cap K_z \cup \{(z,t)\}) - \mathbb{E}\xi((z,t),\mathcal{P}_\lambda \cap K_z)\mathbb{E}\xi(\bar{y},\mathcal{P}_\lambda \cap K_z).$$

The McMullen bound [10] gives

$$|c((z,t),\bar{y};\mathcal{P}_\lambda \cap K_z)| \le C\mathbb{E}[N(\lambda)^d] \le C\lambda^d, \tag{5.5}$$

where here $N(\lambda)$ denotes the cardinality of $\mathcal{P}_\lambda \cap K_z$.

We make the following three modifications to the triple integral (5.4), each one giving an error of $o(1)$:

(i) Replace the integration domain $\{\bar{y} \in K_z\}$ by $\{\bar{y} \in K_z(\epsilon_\lambda^2)\}$. Indeed, uniformly in $\bar{y} \in K_z \setminus K_z(\epsilon_\lambda^2)$ we have

$$\lim_{\lambda\to\infty} \lambda^{1+2\beta}c((z,t),\bar{y};\mathcal{P}_\lambda \cap K_z) = 0,$$

by Lemma 4.1(a), the bound (5.5), and the Cauchy–Schwarz inequality. Since

$$\sup_{\lambda\ge 1} \sup_{(z,t),\bar{y}\in K_z\setminus K_z(\epsilon_\lambda^2)} \lambda^{1+2\beta}c((z,t),\bar{y};\mathcal{P}_\lambda \cap K_z) \le C,$$

the assertion follow by the bounded convergence theorem.

(ii) Replace the integration domain $\{\bar{y} \in K_z(\epsilon_\lambda^2)\}$ by $\{\bar{y} \in K_z(\epsilon_\lambda^2) \cap B_{2D_1\epsilon_\lambda}((z,t))\}$ (use Lemma 4.9 and the bounded convergence theorem).

(iii) Replace the integration domain $[0, t(z)]$ by $[0, \epsilon_\lambda^2]$, as at (5.1). These modifications yield

$$I_2(\lambda) = \lambda^{1+2\beta} \int_{z\in\partial K} \int_0^{\epsilon_\lambda^2} \int_{\bar{y}\in K_z(\epsilon_\lambda^2)\cap B_{2D_1\epsilon_\lambda}((z,t))} g((z,t))g(\bar{y})c((z,t),\bar{y};\mathcal{P}_\lambda \cap K_z)d\bar{y} \times \Pi_{i=1}^{d-1}(1 - tC_{z,i})dtdz + o(1). \tag{5.6}$$

Changing variables with $\bar{y} = (r, u)$ gives $d\bar{y} = r^{d-1} dr d\sigma_{d-1}(u)$ and it also gives

$$\begin{aligned} \mathcal{T}^{\lambda,z}((r,u)) &= \left(\left(r_z^d \lambda \right)^{\beta} \exp_{d-1}^{-1}(u), \left(r_z^d \lambda \right)^{2\beta} \left(1 - \frac{r}{r_z} \right) \right) \\ &= \left(\left(r_z^d \lambda \right)^{\beta} v, h' \right) = (v', h') = w'. \end{aligned}$$

Thus the covariance $c((z,t), \bar{y}; \mathcal{P}_\lambda \cap K_z)$ transforms to $c^{\lambda,z}((\mathbf{0}, h), (v', h'); \mathcal{P}^{\lambda,z})$, with $c^{\lambda,z}$ defined at (4.18). Now change variables with $t = r_z (r_z^d \lambda)^{-2\beta} h$, $v' = (r_z^d \lambda)^{\beta} v$, and $h' = (r_z^d \lambda)^{2\beta} (1 - \frac{r}{r_z})$.

The differential $\lambda^{1+2\beta} \Pi_{i=1}^{d-1} (1 - t C_{z,i}) r^{d-1} dr d\sigma_{d-1}(u) dt dz$ transforms to the differential

$$\begin{aligned} &\lambda^{1+2\beta} \Pi_{i=1}^{d-1} (1 - r_z (r_z^d \lambda)^{-2\beta} h C_{z,i}) ((1 - (r_z^d \lambda)^{-2\beta} h') r_z)^{d-1} r_z (r_z^d \lambda)^{-2\beta} dh' \\ &\quad \times (r_z^d \lambda)^{-\beta(d-1)} dv' r_z (r_z^d \lambda)^{-2\beta} dh dz \\ &\quad = \Pi_{i=1}^{d-1} (1 - r_z (r_z^d \lambda)^{-2\beta} h C_{z,i}) (1 - (r_z^d \lambda)^{-2\beta} h')^{d-1} r_z^{1-2\beta d} dh' dv' dh dz. \end{aligned}$$

The upper limit of integration ϵ_λ^2 in (5.6) changes to $h(\lambda, z)$ and the domain of integration $K_z(\epsilon_\lambda^2) \cap B_{2D_1 \epsilon_\lambda}((z,t))$ gets mapped to $S^{\lambda,z}$. This gives

$$I_2(\lambda) = \int\limits_{z \in \partial K} \int\limits_0^{h(\lambda,z)} \int\limits_{(v',h') \in S^{\lambda,z}} G_\lambda(h', v', h, z) dh' dv' dh dz + o(1), \tag{5.7}$$

where, recalling $r_z^{1-2\beta d} = \kappa(z)^{1/(d+1)}$, we get

$$\begin{aligned} G_\lambda(h', v', h, z) &:= \kappa(z)^{1/(d+1)} g((z, o(1))) g(r_z(1 - o(1)), (r_z^d \lambda)^{-\beta} v') \\ &\quad \times c^{\lambda,z}((\mathbf{0}, h), (v', h'); \mathcal{P}^{\lambda,z}) \Pi_{i=1}^{d-1} (1 - o(1))(1 - o(1))^{d-1}, \end{aligned}$$

where the first $o(1)$ denotes a term tending to zero uniformly in $(z, h) \in \partial K \times [0, h(\lambda, z)]$, the second denotes a term tending to zero uniformly in r_z, $z \in \partial K$, and the latter two $o(1)$ terms denote a quantity tending to zero uniformly in $(v', h') \in S^{\lambda,z}$. We next restrict the integration domain $S^{\lambda,z}$ to $S^{\lambda,z} \cap B^{\lambda,z}$ since by Lemma 4.4 and the moment bounds (4.9) we have

$$\int\limits_{z \in \partial K} \int\limits_0^{h(\lambda,z)} \int\limits_{(v',h') \in S^{\lambda,z} \cap (B^{\lambda,z})^c} G_\lambda(h', v', h, z) dh' dv' dh dz = o(1).$$

By Lemma 4.8, uniformly on the range $\{(v', h') \in S^{\lambda,z} \cap B^{\lambda,z}\}$ and uniformly over $h \in [0, h(\lambda, z)]$, the covariance term $c^{\lambda,z}((\mathbf{0}, h), (v', h'); \mathcal{P}^{\lambda,z})$ differs from the covariance term $c^{\lambda,z}((\mathbf{0},h),(v',h');\mathcal{P}^{\lambda,z}_{r_z})$ by a term of order $\lambda^{-\beta/3}$, modulo logarithmic terms. The integral of this difference over

$$(h',v',h,z)\in S^{\lambda,z}\times[0,h(\lambda,z)]\times\partial K$$

is also $o(1)$. Recalling (5.7), this gives

$$I_2(\lambda)=\int_{z\in\partial K}\int_{|h|\le h(\lambda,z)}\int_{(v',h')\in S^{\lambda,z}\cap B^{\lambda,z}}\tilde{G}_\lambda(h',v',h,z)dh'dv'dhdz+o(1),$$

where

$$\tilde{G}_\lambda(h',v',h,z)=\kappa(z)^{1/(d+1)}g((z,o(1)))g(r_z(1-o(1)),o(1))\\ \times c^{\lambda,z}((\mathbf{0},h),(v',h');\mathcal{P}^{\lambda,z}_{r_z})\Pi_{i=1}^{d-1}(1-o(1))(1-o(1))^{d-1}.$$

Recalling the definition of $\zeta_{\xi^{(\infty)}}$ at (2.2) we get via Lemma 7.2 of [8] that

$$\lim_{\lambda\to\infty}\tilde{G}_\lambda(h',v',h,z)=\kappa(z)^{1/(d+1)}g(z)^2\,\zeta_{\xi^{(\infty)}}((\mathbf{0},h),(v',h');\mathcal{P}).$$

The first part of Lemma 7.3 of [8] shows that $c^{\lambda,z}((\mathbf{0},h),(v',h');\mathcal{P}^{\lambda,z}_{r_z})$ is dominated by an integrable function of h', v', h and z on $[0,\infty)\times\mathbb{R}^{d-1}\times[0,\infty)\times\partial K$. Since $\sup_{z\in\partial K}|r_z^{d+1}|$ and $||g||_\infty$ are both bounded and since the integration domain $S^{\lambda,z}\cap B^{\lambda,z}$ increases up to $\mathbb{R}^{d-1}\times[0,\infty)$, the dominated convergence theorem gives

$$\lim_{\lambda\to\infty}I_2(\lambda)=\int_{z\in\partial K}g(z)^2\kappa(z)^{1/(d+1)}\int_0^\infty\int_{\mathbb{R}^{d-1}}\int_0^\infty\zeta_{\xi^{(\infty)}}((\mathbf{0},h),(v',h');\mathcal{P})dh'dv'dhdz. \tag{5.8}$$

Combining (5.3) and (5.8) gives

$$\begin{aligned}\lim_{\lambda\to\infty}&\lambda^{-1+2\beta}\mathrm{Var}[\langle g,\mu^\xi_\lambda\rangle]\\ &=\int_{\partial K}g(z)^2\kappa(z)^{1/(d+1)}\int_0^\infty\mathbb{E}(\xi^{(\infty)}((\mathbf{0},h),\mathcal{P}))^2dh\;+\\ &\quad+\kappa(z)^{1/(d+1)}\int_0^\infty\int_{\mathbb{R}^{d-1}}\int_0^\infty\zeta_{\xi^{(\infty)}}((\mathbf{0},h),(v',h');\mathcal{P})dh'dv'dhdz.\end{aligned}$$

Recalling the definition of $\sigma^2(\xi^{(\infty)})$ at (2.3), this yields

$$\lim_{\lambda\to\infty}\lambda^{-1+2\beta}\mathrm{Var}[\langle g,\mu^\xi_\lambda\rangle]=\sigma^2(\xi^{(\infty)})\int_{\partial K}g(z)^2\kappa(z)^{1/(d+1)}dz.$$

This concludes the proof of variance asymptotics and the proof of Theorem 2.1. $\square$

Remark If one could show that $\xi^{\lambda,z}$ localize in the sense of (4.6), then one could show that the moment bounds of Lemma 4.3 are independent of λ. We expect that our main results would then hold for $K \in \mathcal{K}_+^2$ by making these three changes: (i) replace the right-hand side of (4.10) with $o(1)|v'|^2$, (ii) in Lemmas 4.5 and 4.8, drop the restrictions $w_0', w' \in S^{\lambda,z} \cap B^{\lambda,z}$, and replace the bounds on the right-hand side of (4.11) and (4.19) with $o(1)$ bounds, and (iii) show that $c^{\lambda,z}((\mathbf{0}, h), (v', h'); \mathcal{P}^{\lambda,z})$ decays exponentially in $|v'|$ and h', showing that $G_\lambda(h', v', h, z)$ is integrable. We could then directly apply the dominated convergence theorem to $\mathbb{E}\xi^{\lambda,z}((\mathbf{0}, h), \mathcal{P}^{\lambda,z})$ and $c^{\lambda,z}((\mathbf{0}, h), (v', h'); \mathcal{P}^{\lambda,z})$ without needing the error approximations of Lemmas 4.5 and 4.8.

6 Proof of Theorem 1.2

Fix $K \in \mathcal{K}_+^3$. The image of K by $x \longmapsto \mathrm{vol}(K)^{-1/d} \cdot x$ is a convex body of unit volume so without loss of generality, we may assume in this section that $\mathrm{vol}(K) = 1$. The proof of Theorem 1.2 via Theorem 1.1 is a rewriting of a result previously obtained by Vu (see [20], Proposition 8.1) in the case $k = 0$. For sake of completeness, we include here a proof which does not use large deviation results for $f_k(K_\lambda)$.

The method here uses a coupling of the Poisson point process of intensity n and the binomial point process. Moreover, whether one uses de-Poissonization based on coupling or the approach of [20], the methods in either case establish that much of the limit theory of [8] extends to binomial input. This addresses a technical issue raised at the end of Section 1 of [8].

Let X_i, $i \geq 1$, be a sequence of i.i.d. uniform random variables in $K(\epsilon_n^2)$ and put $\mathcal{X}_n := \{X_1, \dots, X_n\}$. For sake of simplicity, we denote by $f_k(\mathcal{X}_n \cap K(\epsilon_n^2))$ the number of k-dimensional faces of the convex hull of $\mathcal{X}_n \cap K(\epsilon_n^2)$. In particular, we have

$$f_k(\mathcal{X}_n \cap K(\epsilon_n^2)) := \sum_{X_i \in \mathcal{X}_n \cap K(\epsilon_n^2)} \xi_k(X_i, \mathcal{X}_n \cap K(\epsilon_n^2)).$$

We start with two preliminary lemmas which describe the growth of $f_k(\mathcal{X}_n \cap K(\epsilon_n^2))$.

Lemma 6.1 *Fix $K \in \mathcal{K}_+^3$. For all $k \in \{0, 1, \dots, d-1\}$ there is an event $F(n)$, $P[F(n)^c] = O(n^{-4d})$, and a constant $C_1 \in (0, \infty)$ such that on $F(n)$*

$$|f_k(\mathcal{X}_n \cap K(\epsilon_n^2)) - f_k(\mathcal{X}_{n+1} \cap K(\epsilon_n^2))| \leq C_1 (\log n)^{k+1}. \tag{6.1}$$

Proof As in the proof of Lemma 4.1 and as on the pages 499–502 of [14], there is an event $F_1(n)$ with $P[F_1(n)^c] = O(n^{-4d})$, such that on $F_1(n)$ we have for $X_i \in K(\epsilon_n^2)$, $1 \leq i \leq n+1$,

$$\xi_k(X_i, \mathcal{X}_n \cap K(\epsilon_n^2)) = \xi_k(X_i, \mathcal{X}_n \cap K(\epsilon_n^2) \cap B_{D_1 \epsilon_n}(X_i)).$$

It follows that if $X_i \in B^c_{D_1\epsilon_n}(X_{n+1}) \cap K(\epsilon_n^2)$, then on $F_1(n)$ we have

$$\xi_k(X_i, \mathcal{X}_n \cap K(\epsilon_n^2)) = \xi_k(X_i, \mathcal{X}_{n+1} \cap K(\epsilon_n^2)).$$

Thus on $F_1(n)$ we have

$$\begin{aligned}&|f_k(\mathcal{X}_n \cap K(\epsilon_n^2)) - f_k(\mathcal{X}_{n+1} \cap K(\epsilon_n^2))|\\ &\quad \le \xi_k(X_{n+1}, \mathcal{X}_{n+1}) + \sum_{X_i \in B_{D_1\epsilon_n}(X_{n+1})\cap K(\epsilon_n^2)} |\xi_k(X_i, \mathcal{X}_n \cap K(\epsilon_n^2))\\ &\qquad - \xi_k(X_i, \mathcal{X}_{n+1} \cap K(\epsilon_n^2))|.\end{aligned}$$

The Lebesgue measure of $B_{D_1\epsilon_n}(X_{n+1}) \cap K(\epsilon_n^2)$ is $O(\epsilon_n^{d-1}\epsilon_n^2) = O(\epsilon_n^{d+1}) = O(\log n/n)$. There is thus an event $F_2(n)$, with $P[F_2^c(n)] = O(n^{-4d})$, such that on $F_2(n)$ we have

$$\text{card}\{\mathcal{X}_n \cap B_{D_1\epsilon_n}(X_{n+1}) \cap K(\epsilon_n^2)\} = O(\log n).$$

The proof of Lemma 4.3 shows that for $X_i \in B_{D_1\epsilon_n}(X_{n+1}) \cap K(\epsilon_n^2)$ there is an event $F_3(n)$, $P[F_3(n)^c] = O(n^{-4d})$, such that on $F_3(n)$ we have

$$\xi_k(X_i, \mathcal{X}_n) = O((\log n)^k).$$

The same occurs for $\xi_k(X_{n+1}, \mathcal{X}_{n+1})$. On the event $F(n) := F_1(n) \cap F_2(n) \cap F_3(n)$ we get (6.1), concluding the proof of Lemma 6.1. □

Lemma 6.2 *For all $k \in \{0, 1, \ldots, d-1\}$ there is a constant C_2 such that for all integers $l = 1, 2, \ldots, n$ we have*

$$P[|f_k(\mathcal{X}_n \cap K(\epsilon_n^2)) - f_k(\mathcal{X}_{n+l} \cap K(\epsilon_n^2))| \ge C_2 l(\log n)^{k+1}] \le C_2 l n^{-4d}.$$

Proof We have

$$\begin{aligned}|f_k(\mathcal{X}_n \cap K(\epsilon_n^2)) - f_k(\mathcal{X}_{n+l} \cap K(\epsilon_n^2))| \le \sum_{i=0}^{l-1} &|f_k(\mathcal{X}_{n+i} \cap K(\epsilon_n^2))\\ &- f_k(\mathcal{X}_{n+i+1} \cap K(\epsilon_n^2))|.\end{aligned}$$

By Lemma 6.1, the ith summand is bounded by $C_1(\log(n+i))^{k+1}$ on a set whose complement probability is $O(n^{-4d})$. Since $C_1(\log(n+i))^{k+1} \le C(\log 2n)^{k+1}$, the result follows. □

For every $\lambda > 0$, let $N(\lambda)$ denote a Poisson variable of mean λ and for every integer n and $p \in (0, 1)$, let Bi(n, p) denote a Binomial variable of parameters n and p. The next result yields Theorem 1.2.

Theorem 6.1 *Fix $K \in \mathcal{K}_+^3$ and let K_n' be the convex hull of n independent and uniformly distributed points in K, $n \geq 1$. For all $k \in \{0, 1, \ldots, d-1\}$ we have*

$$|\mathrm{Var} f_k(K_n') - \mathrm{Var} f_k(K_{N(n)}')| = O\left(n^{1-\frac{3}{d+1}+o(1)}\right).$$

Proof For all integers m we put $H_m := f_k(K_m')$. We have

$$\mathrm{Var} H_n = \mathrm{Var} H_{N(n)} + \mathrm{Var}(H_n - H_{N(n)}) + 2\mathrm{Cov}(H_{N(n)}, H_n - H_{N(n)}).$$

By (1.2), we have

$$\begin{aligned} \mathrm{Cov}(H_{N(n)}, H_n - H_{N(n)}) &\leq \sqrt{\mathrm{Var} H_{N(n)}}\, ||H_n - H_{N(n)}||_2 \\ &= O\left(n^{(d-1)/2(d+1)}\right) ||H_n - H_{N(n)}||_2. \end{aligned} \tag{6.2}$$

It is thus enough to show

$$||H_n - H_{N(n)}||_2^2 = O(n^{1-\frac{4}{d+1}+o(1)}) \tag{6.3}$$

since then the last two terms in (6.2) are both $O\left(n^{1-\frac{3}{d+1}+o(1)}\right)$.

Given the binomial and Poisson distributions $\mathcal{L}(\mathrm{Bi(n, \epsilon_n^2)})$ and $\mathcal{L}(N(n\epsilon_n^2))$, there exist coupled random variables $\mathrm{Bi(n, \epsilon_n^2)}$ and $N(n\epsilon_n^2)$ such that

$$P[\mathrm{Bi(n, \epsilon_n^2)} \neq \mathrm{N(n\epsilon_n^2)}] \leq \epsilon_n^2; \tag{6.4}$$

see e.g. (1.4) and (1.23) of [5].

Enumerate the points $\mathcal{P}_n \cap K(\epsilon_n^2)$ by $X_1, X_2, \ldots, X_{N(\epsilon_n^2)}$. Given $\mathrm{Bi(n, \epsilon_n^2)}$, consider the coupled point set $\mathcal{Y}_n$ obtained by discarding or adding i.i.d. points X_i in $K(\epsilon_n^2)$:

$$\mathcal{Y}_n := \begin{cases} X_1, \ldots, X_{N(\epsilon_n^2)-(N(\epsilon_n^2)-\mathrm{Bi(n,\epsilon_n^2)})^+}, & \text{if } N(\epsilon_n^2) \geq \mathrm{Bi(n, \epsilon_n^2)} \\ X_1, \ldots, X_{N(\epsilon_n^2)+(\mathrm{Bi(n,\epsilon_n^2)}-\mathrm{N}(\epsilon_n^2))^+}, & \text{if } N(\epsilon_n^2) < \mathrm{Bi(n, \epsilon_n^2)}. \end{cases}$$

Then $\mathcal{Y}_n \stackrel{\mathcal{D}}{=} \mathcal{X}_n \cap K(\epsilon_n^2) = X_1, X_2, \ldots, X_{\mathrm{Bi(n,\epsilon_n^2)}}$. We use this coupling of the point sets $\mathcal{P}_n \cap K(\epsilon_n^2)$ and $\mathcal{X}_n \cap K(\epsilon_n^2)$ in all that follows.

Denoting the convex hull of m i.i.d. points $X_1, \ldots, X_m$ on $K(s)$ by $K(s)_m'$, we have

$$\begin{aligned} ||H_n - H_{N(n)}||_2^2 &= \int (f_k(K_n') - f_k(K_{N(n)}'))^2 dP \\ &= \int \left[f_k(K(\epsilon_n^2)_{\mathrm{Bi}(n,\mathrm{vol}(K(\epsilon_n^2)))}') - f_k(K(\epsilon_n^2)_{N(n\mathrm{vol}(K(\epsilon_n^2)))}')\right]^2 dP + o(1), \end{aligned}$$

where the last equality follows from the $O(n^{-4d})$ probability bounds of Lemma 4.1(a), the bounds $f_k(K(\epsilon_n^2)_j') \leq C_3 j^{d/2}$, and the Cauchy–Schwarz inequality.

Let $E_n := \{Bi(\mathrm{n}, \mathrm{vol}(K(\epsilon_n^2))) \neq N(\mathrm{nvol}(K(\epsilon_n^2)))\}$ and recall from (6.4) that $P[E_n] \leq \epsilon_n^2$. On E_n^c the integrand vanishes. Thus

$$||H_n - H_{N(n)}||_2^2 = \int \left[f_k(K(\epsilon_n^2)_{\mathrm{Bi(n,vol(K(\epsilon_n^2)))}}) - f_k(K(\epsilon_n^2)_{N(n\mathrm{vol}(K(\epsilon_n^2)))}) \right]^2 \mathbf{1}(E_n) dP + o(1).$$

By the Bernstein inequality there is a constant C_4 such that for all $p \in (0, 1/2)$ we have

$$|\mathrm{Bi}(n, p) - np| \leq C_4 (\log(np)) \sqrt{np}$$

with probability at least $1 - O(n^{-4d})$. By Proposition A.2.3 of [5], and taking C_4 larger if necessary, we also have

$$|\mathrm{N}(np) - np| \leq C_4 (\log(np)) \sqrt{np}$$

with probability at least $1 - O(n^{-4d})$. A modification of Lemma 6.2 shows that there is an event $G_n(1)$ with probability at least $1 - O((\log n)^{1+1/(d+1)} n^{1/2-1/(d+1)-4d})$ such that on $G_n(1)$ we have

$$|f_k(K(\epsilon_n^2)_{\mathrm{Bi(n,vol(K(\epsilon_n^2)))}}) - f_k(K(\epsilon_n^2)_{n\lfloor \mathrm{vol(K(\epsilon_n^2))} \rfloor})|^2 = O((\log n)^{2k+4} n \epsilon_n^2).$$

Similarly, there is an event $G_n(2)$ with probability at least $1 - O((\log n)^{1+1/(d+1)} n^{1/2-1/(d+1)-4d})$ such that on $G_n(2)$ we have

$$|f_k(K(\epsilon_n^2)_{\mathrm{N(nvol(K(\epsilon_n^2)))}}) - f_k(K(\epsilon_n^2)_{n\lfloor \mathrm{vol(K(\epsilon_n^2))} \rfloor})|^2 = O((\log n)^{2k+4} n \epsilon_n^2).$$

On the event $G_n := G_n(1) \cup G_n(2)$ we have

$$|f_k(K(\epsilon_n^2)_{\mathrm{Bi(n,\epsilon_n^2)}}) - f_k(K(\epsilon_n^2)_{\mathrm{N(n\epsilon_n^2)}})|^2 = O((\log n)^{2k+4} n \epsilon_n^2). \tag{6.5}$$

By McMullen's bound [10]

$$|f_k(K(\epsilon_n^2)_{\mathrm{Bi(n,\epsilon_n^2)}}) - f_k(K(\epsilon_n^2)_{\mathrm{N(n\epsilon_n^2)}})|^2 \leq C_3 (\mathrm{Bi(n, \epsilon_n^2)^d + N(n\epsilon_n^2)^d})$$

always holds. It follows by the Cauchy–Schwarz inequality that

$$\int \left[f_k(K(\epsilon_n^2)_{\mathrm{Bi(n,\epsilon_n^2)}}) - f_k(K(\epsilon_n^2)_{N(n\epsilon_n^2)}) \right]^2 \mathbf{1}(E_n) \mathbf{1}(G_n^c) dP = o(1),$$

whence in view of (6.5)

$$||H_n - H_{N(n)}||_2^2 = O\left((\log n)^{2k+4} n \epsilon_n^2 \int \mathbf{1}(E_n) \mathbf{1}(G_n) dP \right) + o(1).$$

It follows that

$$||H_n - H_{N(n)}||_2^2 = O((\log n)^{2k+4} n \epsilon_n^2 P[E_n]) + o(1) = O((\log n)^{2k+4} n \epsilon_n^4) + o(1).$$

This shows (6.3) and concludes the proof of Theorem 6.1. □

Acknowledgments J. Yukich gratefully acknowledges the kind hospitality and support of the Département de Mathématiques at the Université de Rouen, where some of this work was completed.

References

1. Bárány, I.: Random polytopes in smooth convex bodies. Mathematika **39**, 81–92 (1992) [Correction, Mathematika, 51, 31 (2004)]
2. Bárány, I., Fodor, F., Vigh, V.: Intrinsic volumes of inscribed random polytopes in smooth convex bodies. Adv. Appl. Probab. **42**, 605–619 (2010)
3. Bárány, I., Reitzner, M.: Poisson polytopes. Ann. Probab. **38**, 1507–1531 (2010)
4. Bárány, I., Reitzner, M.: The variance of random polytopes. Adv. Math. **225**, 1986–2001 (2010)
5. Barbour, A.D., Holst, L., Janson, S.: Poisson Approximation. Oxford University Press, New York (1992)
6. Böröczky, K., Fodor, F., Hug, D.: The mean width of random polytopes circumscribed around a convex body. J. Lond. Math. Soc. **81**, 499–523 (2010)
7. Buchta, C.: An identity relating moments of functionals of convex hulls. Discrete Comput. Geom. **33**, 125–142 (2005)
8. Calka, P., Schreiber, T., Yukich, J.E.: Brownian limits, local limits, and variance asymptotics for convex hulls in the ball. Ann. Probab. **41**(1), 55–108 (2013)
9. Groeneboom, P.: Limit theorems for convex hulls. Prob. Theory Related Fields **79**, 327–368 (1988)
10. McMullen, P.: The maximum number of faces of a convex polytope. Mathematika **17**, 179–184 (1970)
11. Pardon, J.: Central limit theorems for random polygons in an arbitrary convex set. Ann. Probab. **39**, 881–903 (2011)
12. Reiss, R.-D.: A course on point processes. In: Springer Series in Statistics. Springer, Berlin (1993)
13. Reitzner, M.: Stochastic approximation of smooth convex bodies. Mathematika **51**, 11–29 (2004)
14. Reitzner, M.: Central limit theorems for random polytopes. Probab. Theory Related Fields **133**, 488–507 (2005)
15. Reitzner, M.: The combinatorial structure of random polytopes. Adv. Math. **191**, 178–208 (2005)
16. Rényi, A., Sulanke, R.: Über die konvexe Hülle von n zufällig gewählten Punkten. Z. Wahrscheinlichkeitstheorie Verw. Gebiete **2**, 75–84 (1963)
17. Rényi, A., Sulanke, R.: Über die konvexe Hülle von n zufällig gewählten Punkten II. Z. Wahrscheinlichkeitstheorie Verw. Gebiete **3**, 138–147 (1964)
18. Schneider, R., Weil, W.: Stochastic and Integral Geometry. Springer, Berlin (2008)
19. Schreiber, T., Yukich, J.E.: Variance asymptotics and central limit theorems for generalized growth processes with applications to convex hulls and maximal points. Ann. Probab. **36**, 363–396 (2008)
20. Vu, V.: Central limit theorems for random polytopes in a smooth convex set. Adv. Math. **207**, 221–243 (2006)
21. Weil, W., Wieacker, J.A.: Stochastic geometry. In: Gruber, P.M., Wills, J.M. (eds.) Handbook of Convex Geometry, vol. B, pp. 1391–1438. North-Holland, Amsterdam (1993)