MATH 21, FALL, 2009, FINAL EXAM SOLUTIONS

(1) Find the indicated limits: Show the steps involved. *(5 points/part)*

(a) $\lim_{t\to 1} \frac{t^2-5t+4}{t^2-1}$

Solution:: [Similar to (1a) on the 2008 final]

$$\begin{aligned} \lim_{t\to 1} \frac{t^2-5t+4}{t^2-1} &= \lim_{t\to 1} \frac{(t-1)(t-4)}{(t-1)(t+1)} \\ &= \lim_{t\to 1} \frac{(t-4)}{(t+1)} \\ &= \frac{-3}{2}. \end{aligned}$$

(b) $\lim_{x\to 0} \frac{\arctan x}{e^x-1}$

Solution: [On the review sheet, (1l)] This uses l'Hôpital's Rule, $\frac{0}{0}$ form;

$$\begin{aligned} \lim_{x\to 0} \frac{\arctan x}{e^x-1} &= \lim_{x\to 0} \frac{\left(\frac{1}{1+x^2}\right)}{e^x} \\ &= 1. \end{aligned}$$

(c) $\lim_{x\to\infty} x - \frac{x^2+1}{x+2}$

Solution:: [On the review sheet, (1m)] This should be simplified first,

$$\begin{aligned} \lim_{x\to\infty} x - \frac{x^2+1}{x+2} &= \lim_{x\to\infty} \frac{x(x+2) - \left(x^2+1\right)}{x+2} \\ &= \lim_{x\to\infty} \frac{2x-1}{x+2} \\ &= 2. \end{aligned}$$

(2) Find the indicated derivatives: Show the steps involved. Do not simplify. *(5 points/part)*

(a) $\left((x^2-3)^2(x^3+2x-5)\right)'$

Solution:: [Similar to (2a) on the 2008 final]

$$\begin{aligned} \left((x^2-3)^2(x^3+2x-5)\right)' &= \left((x^2-3)^2\right)'(x^3+2x-5) + (x^2-3)^2(x^3+2x-5)' \\ &= 2(x^2-3)2x(x^3+2x-5) + (x^2-3)^2(3x^2+2) \end{aligned}$$

That is a perfectly acceptable answer. No need to simplify.

(b) $\left(\frac{3x-2}{x+2}\right)'$

Solution:: [Similar to, but easier than, (2b) on the 2008 final]

$$\begin{aligned} \left(\frac{3x-2}{x+2}\right)' &= \frac{(3x-2)'(x+2) - (x+2)'(3x-2)}{(x+2)^2} \\ &= \frac{3(x+2) - 1(3x-2)}{(x+2)^2} \\ &= \frac{8}{(x+2)^2} \end{aligned}$$

(c) $\left(\sinh(x^2)\right)'$

Solution:: [Similar to (2e) on the 2008 final]

$$\big(\sinh(x^2)\big)' = \cosh(x^2)2x.$$

(d) $(\ln(\sec(x) + \tan(x)))'$

Solution:: [Exam #2 review, (1b)] This has an odd answer:

$$\begin{aligned}(\ln(\tan x + \sec x))' &= \frac{(\tan x + \sec x)'}{(\tan x + \sec x)} \\ &= \frac{\sec^2(x) + \sec(x)\tan(x)}{(\tan x + \sec x)} \\ &= \frac{(\tan x + \sec x)\sec(x)}{(\tan x + \sec x)} \\ &= \sec(x).\end{aligned}$$

(e) $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^2} \frac{1}{t^2 + 3t + 1}\,dt\right)$

Solution:: [Similar to review sheet (2f) and 2008 final (2e)] If $F(x) = \int_1^x \frac{1}{t^2+3t+1}\,dt$, then $F'(x) = \frac{1}{x^2+3x+1}$ by the FTC, part 1. So, by the chain rule, what we are asked to find is $\frac{d}{dx}F(x^2)$:

$$\begin{aligned}\frac{d}{dx}\left(\int_1^{x^2} \frac{1}{t^2 + 3t + 1}\,dt\right) &= \frac{d}{dx}F(x^2) \\ &= F'(x^2)2x \\ &= \frac{2x}{x^4 + 3x^2 + 1}.\end{aligned}$$

Some instructors gave a general formula of

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = f(u(x))u'(x) - f(v(x))v'(x),$$

which would be OK, if you used it correctly.

(3) Evaluate the following integrals. *(5 points/part)*

(a) $\displaystyle\int (x^3 - 3x^2 + 2)\,dx$

Solution:: [Similar to 2008 final (3a)]

$$\int (x^3 - 3x^2 + 2)\,dx = \frac{1}{4}x^4 - x^3 + 2x + c.$$

(b) $\displaystyle\int_0^{\pi} \sin(x)\cos(x)\,dx$

Solution:: [Similar to 2008 final (3b) and review sheet (28k), but easier] Make the substitution $u = \sin(x)$. Then $du = \cos(x)dx$, and, when $x = \pi$, $u = 0$ and when $x = 0$, $u = 0$. Hmm. Then, making this substitution,

$$\int_0^{\pi} \sin(x)\cos(x)\,dx = \int_0^0 u du.$$

But since that is the integral from 0 to 0, you don't have to do the integration, which is $\frac{1}{2}u^2$, anyway. The answer is 0.

(c) $\displaystyle\int_{-1}^{1} \frac{x}{1 + x^2}\,dx$

Solution: [Review Sheet (28l)] Substitute $u = 1 + x^2$. Then $du = 2xdx$, when $x = 1$, $u = 2$ and when $x = -1$, $u = 2$ Hmm. So,

$$\begin{aligned} \int_{-1}^{1} \frac{x}{1+x^2}\,dx &= \frac{1}{2}\int_{2}^{2} \frac{du}{u} \\ &= 0. \end{aligned}$$

Here you don't even have to do the integration, since the limits are from 2 to 2. There is no area under that curve, because the interval has length 0. You could to the integration, but it would give the same result. This is one where it is really easier than it would be if you took the trouble to substitute back in terms of x. Alternately, you can see that the original integral is 0 because the integrand is odd, and the original integral is over an interval with 0 at the center.

(d) $\displaystyle\int x\sqrt{x+2}\,dx$

Solution:: [Similar to review sheet (28f), and 2008 final (3d)] This is a tricky u-substitution, $u = x + 2$. Then, of course $du = dx$, but you also substitute for x as $x = u - 2$, so that

$$\begin{aligned} \int x\sqrt{x+2}\,dx &= \int (u-2)\sqrt{u}du \\ &= \int u^{3/2} - 2\sqrt{u}du \\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + c \\ &= \frac{2}{5}(x-1)^{5/2} - \frac{4}{3}(x-1)^{3/2} + c \end{aligned}$$

(e) $\displaystyle\int_{3}^{11} \frac{dx}{\sqrt{2x+3}}$

Solution: [Review sheet (28m)] Substitute $u = 2x + 3$. Then $du = 2dx$, when $x = 3$, $u = 9$ and when $x = 11$, $u = 25$, so

$$\begin{aligned} \int_{3}^{11} \frac{dx}{\sqrt{2x+3}} &= \frac{1}{2}\int_{9}^{25} \frac{du}{\sqrt{u}} \\ &= \sqrt{u}\Big|_{9}^{25} \\ &= 5 - 3 \\ &= 2. \end{aligned}$$

(4) If $x^2 - 3xy + y^2 = 5$, find $\frac{dy}{dx}$ at the point (1,-1). *(10 points).*

Solution: [Review Sheet (6)] Use implicit differentiation, thinking of y as a function of x:

$$\begin{aligned} (5)' &= \left(x^2 - 3xy + y^2\right)' \\ 0 &= 2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} \\ &= 2x - 3y - (3x - 2y)\frac{dy}{dx}, \end{aligned}$$

so

$$\begin{aligned} (3x - 2y)\,\frac{dy}{dx} &= 2x - 3y \\ \frac{dy}{dx} &= \frac{2x - 3y}{3x - 2y}. \end{aligned}$$

At the point (1,-1), $\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{2\cdot 1 - 3(-1)}{3\cdot 1 - 2(-1)} = 1$.

(5) Find all the values of c for which the quartic polynomial $f(x) = x^2 + (x^2 - c)^2$ has each of the following properties. If there are no such values of c write "none". *(15 points)*

Solution: This problem was somewhat different from those on the review materials. However, it is not too bad. If $f(x) = x^2 + (x^2 - c)^2$, then

$$\begin{aligned} f'(x) &= 2x + 2(x^2 - c)2x \\ &= 2x\left(2x^2 + (1 - 2c)\right). \end{aligned}$$

The factor $2x^2 + (1 - 2c)$ has no solutions if $2c < 1$, or $c < 1/2$, one solution if $c = 1/2$, and 2 solutions if $c > 1/2$, so, because of the other solution when $x = 0$:

(a) f has precisely one critical point.

Solution: When $c < 1/2$. Then, $x = 0$ is the only solution. Wait, also, when $c = 1/2$, then $2x^2 + (1 - 2c) = 2x^2$ and 0 is again the only solution.

(b) f has precisely two distinct critical points.

Solution: Never. There are either 3 solutions, or only 1.

(c) f has precisely three distinct critical points.

Solution: When $c > 1/2$, then $x = 0$ and $x = \pm\sqrt{2c - 1}$ are solutions.

(6) (a) State the definition of the derivative of a function $f(x)$ (as a limit): $f'(x) =$ *(5 points)*

Solution: [Review (38), 2008 final (4), and 2007 final (5)] The *definition* of the derivative of a function $f(x)$, as a limit, is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

(b) Using only the definition of the derivative, find the derivative function $f'(x)$ for the function $f(x) = \dfrac{1}{x}$. *(5 points)*

Solution: [Similar to the review (38), 2008 final (4), and 2007 final (5)]

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{x-(x+h)}{(x+h)x}\right)}{h} \\ &= \lim_{h \to 0} \frac{x - (x + h)}{h\,(x + h)\,x} \\ &= \lim_{h \to 0} \frac{-h}{h\,(x + h)\,x} \\ &= \lim_{h \to 0} \frac{-1}{(x + h)\,x} \\ &= -\frac{1}{x^2}. \end{aligned}$$

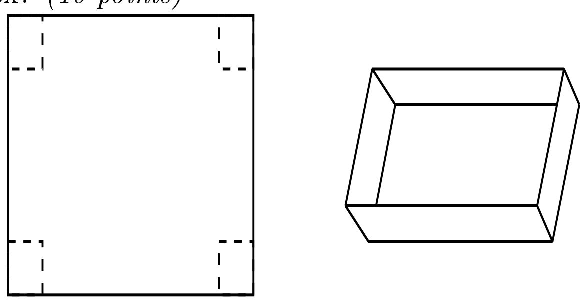
(7) A 20 foot ladder is leaning against a wall. A painter stands on the top of the ladder, minding his own business. Some fool comes by and ties his dog to the base of the ladder, a cat comes along, and the dog chases after the cat, dragging the base of the ladder with him at a rate of 2 feet per second directly away from the wall. How fast is the painter falling when he is 12 feet from the ground? *(10 points)*

Solution: [Review sheet (14)] This is a fairly standard related-rates problem. Set x to be the distance from the base of the ladder to the wall, and set y to be the height of the painter above the ground. We **know** that $\frac{dx}{dt} = +2$ and we **want** to know $\frac{dy}{dt}$ **when** $y = 12$. Then, since the ladder has constant length 20 until it smashes against the ground, the Pythagorean theorem tells us that $x^2 + y^2 = 20^2$, which is an equation between what we know and what we want to know. Differentiate both sides of that equation with respect to time to get the relationship between their rates of change, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. At "the when", you have $y = 12$, so plug into the equation $x^2+y^2 = 20^2$ to see that $x^2+12^2 = 20^2$ at that instant, so $x = 16$ (it's a 3-4-5 triangle). Then, you plug these values into the equation relating the rates, and solve for $\frac{dy}{dt}$:

$$\begin{aligned} 0 &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2 \cdot 16 \cdot 2 + 2 \cdot 12 \cdot \frac{dy}{dt}, \end{aligned}$$

or $\frac{dy}{dt} = -\frac{32}{12} = -\frac{8}{3}$, meaning that the painter is falling at 8/3 feet per second at that instant.

(8) Take a 12 inch by 12 inch rectangle of tin, cut equal squares out of each corner and bend up the sides, then solder the seams on the edges to make a rectangular box with no top. How large a square should you cut out of each corner to maximize the volume of the resulting box? *(10 points)*



Solution:: [2008 Final (7)] If the cut-out corners are all $x \times x$, then the remaining base is a square that is $(12-2x) \times (12-2x)$. Since x is a physical measurement, and you can't cut out more than 1/4 of the whole sheet from each corner, or else you are cutting air instead of the next corner, $0 \le x \le 6$. The volume of the box with the $x \times x$ corners cut out is

$$V = (12-2x)^2 x,$$

and there is no constraint. So, differentiate:

$$\begin{aligned} V' &= (2(12-2x)(-2))\,x + (12-2x)^2 \\ &= (12-2x)(-4x + (12-2x)) \\ &= (12-2x)(-6x+12), \end{aligned}$$

which is zero only when $x = 6$ or $x = 2$. Now we just plug in at all the possible points, $x = 0,\ 2$, or 6, the critical points and endpoints ($x = 6$ counts as both).

$$\begin{aligned} V(0) &= 0, \\ V(2) &= 128, \\ V(6) &= 0. \end{aligned}$$

Clearly $x = 2$ gives the maximum. The question, though, was to find the size of the squares cut out, which would be 2×2.

(9) Find the linear approximation of $f(x) = \sqrt{x+3}$ for x near 1. Your answer should be in the form $g(x) = Ax + B$. *(10 points)*.

Solution:: [Similar to 2008 final (8)] Using linear approximation, $f(x) \approx f(a) + f'(a)(x-a)$, with $f(x) = \sqrt{x+3}$ and $a = 1$ gives, since $f(1) = 2$ and $f'(x) = \frac{1}{2\sqrt{x+3}}$, so

$f'(1) = \frac{1}{4}$,

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ &= 2 + \frac{1}{4}(x-1) \\ &= \frac{1}{4}x + \frac{7}{4}. \end{aligned}$$

(10) The population of certain bacteria grows at a rate proportional to its size. It increases by 60% after 3 days. How long does it take for the population to double? *(10 points)*

Solution: [Review (9)] Since the population of the bacteria grows at a rate proportional to its size, if $P(t)$ is the population, then $P'(t) = kP(t)$, which means that $P(t) = Ae^{kt}$ for some numbers A and k. This is the solution of that general differential equation. To fit this solution to the particular data, we have that $P(0) = A$ (the initial population, which does not need to be a specific number), and $P(3) = 1.6A$, 60% more than we initially had, after 3 days. But then $P(t) = Ae^{kt}$ with the same A as the initial population (at time 0), and

$$\begin{aligned} 1.6A &= P(3) \\ &= Ae^{k3}. \end{aligned}$$

So. $1.6 = e^{3k}$, or, taking natural logs of both sides, $\ln(1.6) = 3k$, or $k = \frac{1}{3}\ln(1.6)$. Then, $P(t)$ is given more explicitly by

$$P(t) = Ae^{\frac{1}{3}\ln(1.6)t}.$$

The time T it takes for the population to double satisfies

$$\begin{aligned} 2A &= P(T) \\ &= Ae^{\frac{1}{3}\ln(1.6)T}, \end{aligned}$$

so, taking ln() of both sides again, $\ln(2) = \ln(e^{\frac{1}{3}\ln(1.6)T}) = \frac{1}{3}\ln(1.6)T$, so $T = \frac{3\ln 2}{\ln(1.6)}$.

(11) Show that the equation $x^3 - x = 3$ has exactly one solution in the interval $1 < x < 2$. You have to show first that some solution exists in the interval, and then show there is only one. State which, if any, theorems you are using to justify your argument. *(10 points)*

Solution: [Review Sheet (21)] Take $f(x) = x^3 - x$. $f(x)$ is continuous, and differentiable, everywhere. Then, since $f(1) = 0$ and $f(2) = 6$, 3 is between $f(1)$ and $f(2)$, so the IVT implies that there is a point $c \in (1,2)$ at which $f(c) = 3$.
Now, assume that there is a second solution, some $a \neq c$ in $(1,2)$ for which $f(a) = 3$ as well. Then, by the MVT, there is a b between a and c, so $b \in (1,2)$, for which

$$\begin{aligned} f'(b) &= \frac{f(c) - f(a)}{c - a} \\ &= \frac{3-3}{c-a} \\ &= 0. \end{aligned}$$

But, on the other hand, $f'(x) = 3x^2 - 1$ for any x, and so $f'(x) = 0$ only when $x = \pm\frac{1}{\sqrt{3}}$. Neither of these points are in $(1,2)$ (since $\frac{1}{\sqrt{3}} < 1$), so such a b cannot exist, which means that a couldn't exist, and so there is only one solution of the equation in $(1,2)$,$1 < x < 2$.

(12) Sketch the graph of the function $f(x) = \dfrac{e^x}{x+1}$ (including everything). Part of this problem is to remember what things you have to find, such as where the curve is increasing or decreasing, et cetera. *(15 points)*

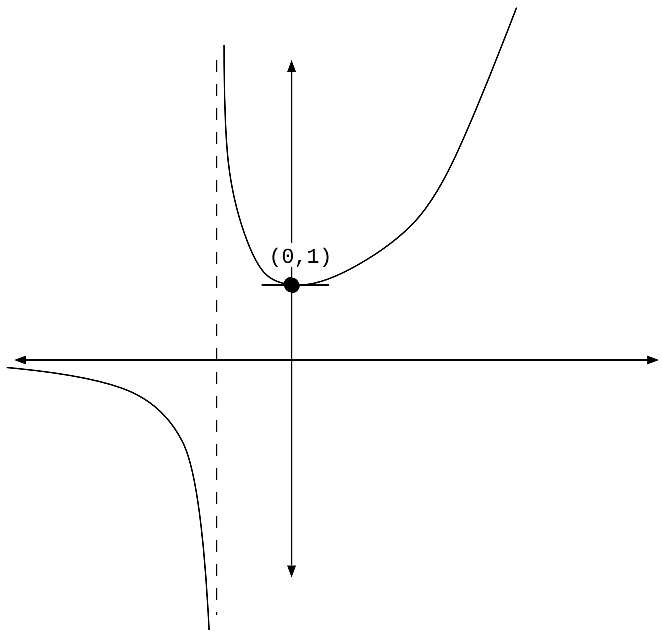
Solution:: [2007 Final (9)] We have to go through the list of things we need to look for:
Domain: Certainly $f(x)$ is defined for all $x \neq -1$.

Asymptotes: There is a vertical asymptote at $x = -1$, and a horizontal asymptote at $y = 0$ for $x \to -\infty$, since $\lim_{x\to-\infty} e^x = 0$.

Intercepts: $f(x)$ is never 0, so there are no x-intercepts. Since $f(0) = 1$, $(0, 1)$ is the y-intercept.

Slope: Since $f'(x) = \frac{e^x((x+1)-1)}{(x+1)^2} = \frac{xe^x}{(x+1)^2}$, $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$ $(x \neq -1)$, and $(0, 1)$ is a local minimum.

Concavity: $f''(x) = \frac{(e^x+xe^x)(x+1)^2-2(x+1)xe^x}{(x+1)^4} = \frac{((x^2+2x+1)-2x)e^x}{(x+1)^3} = \frac{(x^2+1)e^x}{(x+1)^3}$, so the curve is concave up when $x > -1$ and concave down when $x < -1$, and has no inflection points.



Graph:

(13) State both parts of the Fundamental Theorem of Calculus. *(10 points)*

Solution: [Review (47)] Part I: If $f(x)$ is continuous on $[a, b]$, then $g(x) := \int_a^x f(t)dt$ is differentiable on (a, b) and satisfies $g'(x) = f(x)$, so is an antiderivative of $f(x)$. Part II: If $F(x)$ is any antiderivative of a continuous function $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

(14) Where is the function $G(x) = \int_0^x \frac{(t+1)(t+2)}{t^4+1}dt$ increasing, and where is it decreasing? What is $G(0)$? Is $G(-1)$ positive or negative? *(10 points)*

Solution: [Review (51b)] Since, by the FTC, $G'(x) = \frac{(x+1)(x+2)}{x^4+1}$, $G'(x) > 0$ when $x > -1$ or $x < -2$, and $G'(x) < 0$ when $-2 < x < -1$. So, $G(x)$ is increasing on $(-\infty, -2]$ and on $[-1, \infty)$, and $G(x)$ is decreasing on $[-2, -1]$. The endpoints are included for both, since on those intervals the function is increasing, even up to the endpoints. $G(0) = \int_0^0 \frac{(t+1)(t+2)}{t^4+1}dt = 0$ because it's the integral from 0 to 0. $G(x)$ is increasing on $[-1, 0]$, and $G(0) = 0$, so $G(-1)$ has to be negative.

(15) Find the area between the curves $y = x^2 + 2x$ and $y = 3x$. *(15 points)*

Solution: [Review (53)] The curves intersect when $x^2 + 2x = 3x$, or $x = 0$ or $x = 1$. The area is

$$\begin{aligned} A &= \int_0^1 h dx \\ &= \int_0^1 (3x - (x^2 + 2x)) dx \\ &= \int_0^1 x - x^2 dx \\ &= \left. \frac{1}{2}x^2 - \frac{1}{3}x^3 \right|_0^1 \\ &= \frac{1}{6}. \end{aligned}$$