MATH 21, FALL, 2009, PRACTICE EXAM # 1

(1) Find the indicated limits: Show the steps involved.

(a) $\lim_{x\to -4} \dfrac{x^2+5x+4}{x^2+3x-4} =$

Solution: Since both the numerator and denominator go to 0 as $x \to -4$, $(x+4)$ is a factor of both:

$$\begin{aligned}
\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4} &= \lim_{x\to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} \\
&= \lim_{x\to -4} \frac{(x+1)}{(x-1)} \\
&= \frac{(-4+1)}{(-4-1)} \\
&= \frac{-3}{-5} \\
&= \frac{3}{5}.
\end{aligned}$$

(b) $\lim_{x\to\infty} \sqrt{x^2+2x-1} - x =$

Solution: Conjugate:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+2x-1} - x &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x-1} - x\right)\left(\sqrt{x^2+2x-1} + x\right)}{\left(\sqrt{x^2+2x-1} + x\right)} \\
&= \lim_{x\to\infty} \frac{\left(x^2+2x-1\right) - x^2}{\left(\sqrt{x^2+2x-1} + x\right)} \\
&= \lim_{x\to\infty} \frac{2x-1}{\left(\sqrt{x^2+2x-1} + x\right)} \\
&= \lim_{x\to\infty} \frac{(2x-1)\,\frac{1}{x}}{\left(\sqrt{x^2+2x-1} + x\right)\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{\sqrt{\frac{x^2+2x-1}{x^2}} + 1} \\
&= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{\sqrt{1+\frac{2}{x}-\frac{1}{x^2}} + 1} \\
&= \frac{2}{1+1} = 1.
\end{aligned}$$

(c) $\lim_{y\to\infty} \frac{2y^2-5y-3}{5y^2+4y} =$

Solution: Divide out by y^2 top and bottom, and it becomes clear:

$$\begin{aligned}
\lim_{y\to\infty} \frac{2y^2-5y-3}{5y^2+4y} &= \lim_{y\to\infty} \frac{\left(2y^2-5y-3\right)\frac{1}{y^2}}{\left(5y^2+4y\right)\frac{1}{y^2}} \\
&= \lim_{y\to\infty} \frac{\left(2-\frac{5}{y}-\frac{3}{y^2}\right)}{\left(5+\frac{4}{y}\right)} \\
&= \frac{(2-0-0)}{(5+0)} \\
&= \frac{2}{5}.
\end{aligned}$$

(d) $\lim_{t\to 0} \frac{\sqrt{2-t}-\sqrt{2}}{t} =$

Solution: Again, conjugate:

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{2-t}-\sqrt{2}}{t} &= \lim_{t\to 0} \frac{\left(\sqrt{2-t}-\sqrt{2}\right)\left(\sqrt{2-t}+\sqrt{2}\right)}{t\left(\sqrt{2-t}+\sqrt{2}\right)} \\
&= \lim_{t\to 0} \frac{(2-t)-2}{t\left(\sqrt{2-t}+\sqrt{2}\right)} \\
&= \lim_{t\to 0} \frac{-t}{t\left(\sqrt{2-t}+\sqrt{2}\right)} \\
&= \lim_{t\to 0} \frac{-1}{\left(\sqrt{2-t}+\sqrt{2}\right)} \\
&= \frac{-1}{2\sqrt{2}}
\end{aligned}$$

(2) Show that there is a solution of the equation $\cos(x) = \sin(2x)$ in the interval $[0, \pi/2]$.. You can presume that the functions $\cos(x)$ and $\sin(2x)$ are continuous. Explicitly note which theorems you are using.

Solution: This problem deals with the IVT. Find a function that tells you something about this equation. Certainly if $f(x) = \cos(x) - \sin(2x)$, then $f(x) = 0$ whenever the equation $\cos(x) = \sin(2x)$ holds. Also, at the endpoints of the interval, 0 and $\pi/2$, we have

$$f(0) = 1 - 0 = 1$$

and

$$\begin{aligned}
f(\pi/2) &= \cos(\pi/2) - \sin(\pi) \\
&= 0 - 0 \\
&= 0.
\end{aligned}$$

Wait, you don't need the IVT. You were supposed to show that there is a solution of the equation, and the equation is the same as $f(x) = 0$, and here we have $f(\pi/2) = 0$, so sure enough it has a solution. No theorem required.

(3) Find an equation of the tangent line to the curve $y = x^4$ at the point $(2, 16)$.

Solution: The slope of the tangent line is the value of the derivative, at the point $x = 2$. Careful, though, you need the slope at that point, not the derivative at an arbitrary x. The curve is the graph of the function $f(x) = x^4$, so $f'(x) = 4x^3$. At $x = 2$, then, the slope is $4 \cdot 2^3 = 32$. The equation of the tangent line is the equation of the line through the point $(2, 16)$ with this slope, which is the line

$$y - 16 = 32(x - 2),$$

using the point-slope equation of the line. This is a correct answer, and so is

$$y = 32x - 48,$$

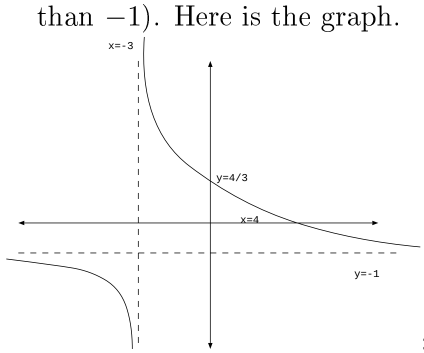
which is the slope-intercept form of the same line.

(4) For the function $f(x) = \dfrac{4-x}{3+x}$, find the vertical and horizontal asymptotes. Use this information to sketch a graph. Does this function have an inverse? (Justify your answer.)

Solution: The graph has a vertical asymptote at $x = -3$ (where the denominator vanishes), and a horizontal asymptote at $y = -1$, the limit at $\pm\infty$. Also, the one-sided limits at -3 are

$$\lim_{x \to -3^+} \frac{4-x}{3+x} = +\infty \text{ and}$$
$$\lim_{x \to -3^-} \frac{4-x}{3+x} = -\infty,$$

which gives a bit more information about the graph. As you approach -3 from the left ($x < -3$), the graph heads down to $-\infty$, and as you approach -3 from the right ($x > -3$), the graph heads to $+\infty$. As a final bit of detail, for x very large, although $\frac{4-x}{3+x}$ is near -1, but is slightly larger than -1, since the numerator is a smaller negative number than the opposite of the denominator. The absolute value will be less than 1, so the fraction is a bit larger than -1, so the graph approaches the horizontal asymptote from above the line $y = -1$. Similarly, on the other side, for x very negative, $f(x)$ is a bit smaller than -1 (larger in absolute value, or "negativer" than -1). Here is the graph.



:

(5) Show that, for the function $f(x) = x\left(\sin\left(\dfrac{1}{x}\right)\right)^2$ the limit $\lim_{x \to 0} f(x)$ exists and is 0.

Solution: This uses the squeeze theorem, but it is a bit subtle. I'm going to split the limit up into the limit from the right, then from the left. From the right, that is, for $x > 0$, you know that

$$0 \leq f(x) \leq x$$

because $0 \leq \left(\sin\left(\frac{1}{x}\right)\right)^2 \leq 1$, so $x \cdot 0 \leq x\left(\sin\left(\frac{1}{x}\right)\right)^2 \leq x$ (remember, $x > 0$). Now, also,

$$\begin{aligned}
\lim_{x\to 0^+} 0 &= 0 \text{ (of course), and} \\
\lim_{x\to 0^+} x &= 0,
\end{aligned}$$

so, by the squeeze theorem,

$$\lim_{x\to 0^+} f(x) = 0$$

since it is trapped between two functions which are headed to 0.

Now, why do we need to deal separately with $x < 0$? Primarily, because, for $x < 0$,

$$0 \geq \quad f(x) \quad \geq x,$$

that is,

$$x \leq \quad f(x) \quad \leq 0.$$

Still,

$$\begin{aligned}
\lim_{x\to 0^-} 0 &= 0 \text{ (of course), and} \\
\lim_{x\to 0^-} x &= 0,
\end{aligned}$$

so, by the squeeze theorem,

$$\lim_{x\to 0^-} f(x) = 0.$$

Finally, since

$$\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0^+} f(x),$$

$$\lim_{x\to 0} f(x) = 0,$$

which is what we needed.

(6)

(a) State the definition of the derivative of a function $f(x)$ (as a limit):
$f'(x) =$

Solution: The *definition* of the derivative of a function $f(x)$, as a limit, is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

(b) Use this definition to determine $f'(x)$, for the function $f(x) = \frac{1}{x+1}$.

Solution:

$$\begin{aligned}
\left(\frac{1}{x+1}\right)' &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ with } f(x)=\frac{1}{x+1}, \text{ so}\\
&= \lim_{h\to 0}\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h}\\
&= \lim_{h\to 0}\frac{\left(\frac{(x+1)-(x+h+1)}{(x+1)(x+h+1)}\right)}{h}\\
&= \lim_{h\to 0}\frac{(x+1)-(x+h+1)}{h\,(x+1)\,(x+h+1)}\\
&= \lim_{h\to 0}\frac{-h}{h\,(x+1)\,(x+h+1)}\\
&= \lim_{h\to 0}\frac{-1}{1\,(x+1)\,(x+h+1)}\\
&= \frac{-1}{(x+1)^2}.
\end{aligned}$$

(7) Find the following derivatives, using the rules we have discussed in class.

(a) $(4x^3-x^2+1)''$

Solution: This is the second derivative, the derivative of the derivative.

$$\begin{aligned}
(4x^3-x^2+1)'' &= (12x^2-2x)'\\
&= 24x-2.
\end{aligned}$$

(b) $\left(\frac{2x+3}{(5x-1)^2}\right)' =$

Solution: Quotient rule:

$$\left(\frac{2x+3}{(5x-1)^2}\right)' = \frac{(2x+3)'\,(5x-1)^2-\left((5x-1)^2\right)'(2x+3)}{(5x-1)^4}.$$

Then we use the chain rule for the second derivative on the right,

$$\begin{aligned}
\left(\frac{2x+3}{(5x-1)^2}\right)' &= \frac{(2x+3)'\,(5x-1)^2-\left((5x-1)^2\right)'(2x+3)}{(5x-1)^4}\\
&= \frac{(2)\,(5x-1)^2-(2\,(5x-1)\,5)\,(2x+3)}{(5x-1)^4}\\
&= \frac{2\,(5x-1)^2-10\,(5x-1)\,(2x+3)}{(5x-1)^4}.
\end{aligned}$$

Of course, as usual, you don't simplify beyond this point.

(c) $\left(e^x\left(x^3+4x\right)\right)' =$

Solution:

$$\begin{aligned}
\left(e^x\left(x^3+4x\right)\right)' &= (e^x)'\left(x^3+4x\right)+e^x\left(x^3+4x\right)'\\
&= (e^x)\left(x^3+4x\right)+e^x\left(3x^2+4\right).
\end{aligned}$$

(d) If $f(x) = \dfrac{\cos(x)}{1+2\sin(x)}$, then $f'(0) =$

Solution:

$$\begin{aligned} f'(x) &= \frac{(\cos(x))'(1+2\sin(x)) - (1+2\sin(x))'(\cos(x))}{(1+2\sin(x))^2} \\ &= \frac{-\sin(x)(1+2\sin(x)) - 2\cos(x)(\cos(x))}{(1+2\sin(x))^2}, \end{aligned}$$

so

$$\begin{aligned} f'(0) &= \frac{-\sin(0)(1+2\sin(0)) - 2\cos(0)(\cos(0))}{(1+2\sin(0))^2} \\ &= \frac{-2}{(1)^2} \\ &= -2. \end{aligned}$$

(e)

$\left(\sin^2(x)+\cos^2(x)\right)' =$

Solution: Since $\sin^2(x)+\cos^2(x) = 1$, $\left(\sin^2(x)+\cos^2(x)\right)' = 0$. You could compute this using the chain rule as well, but this is quicker.

(f) $\left((x^2+2x-3)(x^3-5)\right)' =$

Solution:

$$\begin{aligned} \left((x^2+2x-3)(x^3-5)\right)' &= (x^2+2x-3)'(x^3-5) + (x^2+2x-3)(x^3-5)' \\ &= (2x+2)(x^3-5) + (x^2+2x-3)(3x^2) \end{aligned}$$

(8) Let

$$f(x) := \begin{cases} x^2, & \text{if } x \le 2 \\ mx+b, & \text{if } x > 2 \end{cases}.$$

Find the values of m and b for which the function f will be differentiable everywhere.

Solution: The only issue is at $x = 2$. At all other points, the curve is either a parabola or a straight line, and so differentiable. First, you need to arrange it so that the function will be continuous. That can be arranged by making sure that

$$\lim_{x\to 2^+} f(x) \;=\; \lim_{x\to 2^-} f(x) = f(2).$$

But, $f(2) = 4$,

$$\lim_{x\to 2^-} f(x) \;=\; 4,$$

and

$$\lim_{x\to 2^+} f(x) \;=\; 2m+b.$$

So, these conditions will hold when

$$2m+b = 4.$$

Now, to get the function to be differentiable, the easy way is to make sure that

$$\lim_{x\to 2^+} f'(x) \;=\; \lim_{x\to 2^-} f'(x),$$

or, because $f'(x) = m$ for $x > 2$ and $f'(x) = 2x$ for $x < 2$.

$$\lim_{x\to 2^+} m = \lim_{x\to 2^-} 2x = 4,$$

so $m = 4$ and so $b = -4$.

(9) Show that

$$\lim_{x\to 3} \left(x^2 + x - 4\right) = 8$$

by using an $\epsilon - \delta$ argument.

Solution: If you look first at scratch work, you need to make $\left|(x^2 + x - 4) - 8\right|$ small just by taking x near 3. But

$$\begin{aligned}\left|(x^2 + x - 4) - 8\right| &= \left|x^2 + x - 12\right| \\ &= \left|(x-3)(x+4)\right| \\ &= |x-3|\,|x+4|\,.\end{aligned}$$

Now, $|x-3|$ can be made as small as we want, but we need to also control $|x+4|$. So, we start with an initial estimate on x — remember, all we can do is control how close to 3 x is. So, initially let's say $|x-3| < 1$, which puts x in the range: $-1 < x-3 < 1$ or $2 < x < 4$. Since x is in that range, $6 < x+4 < 8$, so, no matter what, $|x+4| < 8$. Now, look back up at the work we did for $\left|(x^2 + x - 4) - 8\right|$. We need to keep x in the range $2 < x < 4$ so this estimate will work, and we need $|x-3|$ small enough so that, even when multiplied by as big as $|x+4|$ can be (which is 8), the value of $\left|(x^2 + x - 4) - 8\right|$ is less than ϵ. So, $|x-3|$ has to be < 1 *and also* $< \epsilon/8$.
Now, to the proof.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \min\left\{1, \frac{\epsilon}{8}\right\}$. Then, whenever $0 < |x-3| < \delta$,

$$\begin{aligned}\left|(x^2 + x - 4) - 8\right| &= \left|x^2 + x - 12\right| \\ &= \left|(x-3)(x+4)\right| \\ &= |x-3|\,|x+4| \\ &< |x-3|\,8 \\ &< \frac{\epsilon}{8}8 = \epsilon.\end{aligned}$$

□

(10) A ball is tossed up in the air so that its height above the ground t seconds after being tossed is $s(t) = -16t^2 + 32t + 5$ feet.

(a) How fast was the ball moving at the instant when it was tossed?
Solution: The velocity is $v(t) = s'(t) = -32t + 32$. At $t = 0$, the instant the ball was tossed, $v(0) = 32$.

(b) How high was the ball above the ground one second after it was tossed?
Solution: This would be the height at $t = 1$, $s(1) = -16 + 32 + 5 = 21$.

(c) What was its instantaneous velocity at $t = 1$?
Solution: This is of course $v(1) = -32 \cdot 1 + 32 = 0$.