MATH 21, FALL, 2009, EXAM # 2 SOLUTIONS

(1) Find the indicated derivatives. *(5 points/part)*

(a) $(\cosh(2x+3))' =$

Solution: [On review sheet] Using the chain rule, and the derivative of $\cosh(x)$,

$$\begin{aligned}(\cosh(2x+3))' &= \sinh(2x+3)2 \\ &= 2\sinh(2x+3).\end{aligned}$$

(b) $\left(x^2\ln(x)\right)' =$

Solution: [On review sheet] Start with the product rule on this one.

$$\begin{aligned}\left(x^2\ln(x)\right)' &= 2x\ln(x) + x^2\frac{1}{x} \\ &= 2x\ln(x) + x.\end{aligned}$$

(c) $\left(\sin^{-1}(x^2)\right)' =$ *(remember, $\sin^{-1}(x)$ is the inverse sine of x, or $\arcsin(x)$)*

Solution:: [On review sheet]

$$\begin{aligned}\left(\sin^{-1}(x^2)\right)' &= \frac{1}{\sqrt{1-(x^2)^2}}(2x) \\ &= \frac{2x}{\sqrt{1-x^4}}.\end{aligned}$$

(2) Find the following limits. Justify each step. *(5 points/part)*

(a) $\lim_{x\to 0}\frac{\tan(2x)}{\sinh(5x)} =$

Solution: [Similar to 37e of review sheet] Apply l'Hôpital's rule, since this is of the form $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(2x)}{\sinh(5x)} &= \lim_{x\to 0}\frac{2\sec^2(2x)}{5\cosh(5x)} \\ &= \frac{2}{5}.\end{aligned}$$

(b) $\lim_{x\to\infty} x\left(e^{1/x}-1\right) =$

Solution: [4b on 1996 exam] This is an indeterminant form, but of the sort $\infty\cdot 0$. So, invert the x and put it in the denominator, to make it amenable to l'Hôpital's rule.

$$\begin{aligned}\lim_{x\to\infty} x\left(e^{1/x}-1\right) &= \lim_{x\to\infty}\frac{\left(e^{1/x}-1\right)}{\left(\frac{1}{x}\right)} \\ &= \lim_{x\to\infty}\frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)} \\ &= \lim_{x\to\infty} e^{1/x} \\ &= 1.\end{aligned}$$

(c) $\lim_{x \to 0} \dfrac{\sin(x) - x}{x^3} =$

Solution: [Similar to 37g on review sheet] Here you have to apply l'Hôpital's rule several times, since even after applying it twice it is still of the form $\frac{0}{0}$.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin(x) - x}{x^3} &= \lim_{x \to 0} \frac{\cos(x) - 1}{3x^2} \\
&= \lim_{x \to 0} \frac{-\sin(x)}{6x}, \text{ still of form } \frac{0}{0}, \\
&= \lim_{x \to 0} \frac{-\cos(x)}{6} \\
&= -\frac{1}{6}.
\end{aligned}$$

(3) Find the points on the curve

$$x^2 + 2xy + 4y^2 = 3$$

where the curve is horizontal, that is, where $dy/dx = 0$. *(10 points)*

Solution: Use implicit differentiation to find dy/dx:

$$\begin{aligned}
x^2 + 2xy + 4y^2 &= 3 \\
2x + 2y + 2x\frac{dy}{dx} + 8y\frac{dy}{dx} &= 0 \\
2x + 2y + (2x + 8y)\frac{dy}{dx} &= 0 \\
+(2x + 8y)\frac{dy}{dx} &= -(2x + 2y) \\
\frac{dy}{dx} &= -\frac{(2x + 2y)}{(2x + 8y)},
\end{aligned}$$

Then, set that to 0 to answer the question:

$$\begin{aligned}
0 &= -\frac{(2x + 2y)}{(2x + 8y)} \iff \\
0 &= 2x + 2y \iff \\
y &= -x,
\end{aligned}$$

then figure out what points on the curve $x^2 + 2xy + 4y^2 = 3$ satisfy $y = -x$.

$$\begin{aligned}
x^2 + 2xy + 4y^2 &= 3 \\
x^2 - 2xx + 4(-x)^2 &= 3 \\
3x^2 &= 3,
\end{aligned}$$

so $x = \pm 1$. So, the points where the curve is horizontal are the points $(1, -1)$ and $(-1, 1)$.

(4) During the early stages of a disease outbreak, the rate of increase of the number of infected people in a community is proportional to the number who are already infected. If, three weeks ago, there were 30 people on Lehigh's campus infected with the H1N1 flu, and, two weeks ago there were 100, how many people are there now who are infected with the H1N1 flu? *(10 points)*

Solution: [Similar to 7 and 8 on the review sheet, and 7 on the 2008 exam] Set $t = 0$ three weeks ago, at your first data point. Since we assume that the population $P(t)$ of infected individuals satisfies $P' = kP$ (rate of increase is proportional to the population), necessarily $P = Ae^{kt}$ for some constants A and k. But, you have $P(0) = 30$, so

$$30 = P(0) = Ae^0 = A.$$

So, $P(t) = 30e^{kt}$. You also have

$$\begin{aligned} 100 &= P(1) \\ &= 30e^{k \cdot 1}, \text{ or} \\ \frac{100}{30} &= e^k \\ k &= \ln\left(\frac{10}{3}\right), \end{aligned}$$

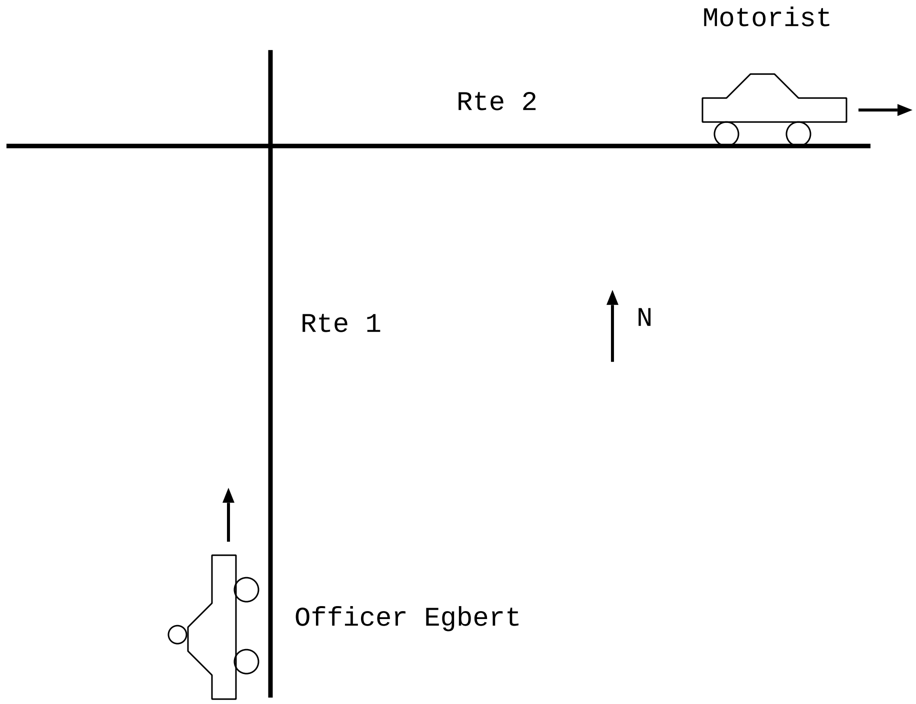
so $P = 30e^{\ln(10/3)t} = 30\left(\frac{10}{3}\right)^t$. The question then is to find the number currently infected, which would be $P(3)$.

$$\begin{aligned} P(3) &= 30\left(\frac{10}{3}\right)^3 \\ &= \frac{30,000}{27} \\ &= \frac{10,000}{9} \\ &= 1,111\frac{1}{9}, \end{aligned}$$

so, to the nearest whole person, there would be 1,111 people currently infected.

(5) Officer Egbert is traveling due North on Route 1, moving at 60 m.p.h. He spies a motorist driving East on Route 2. When he is 0.4 miles South of the intersection with Route 2 (which heads due East and West, at a right angle to Route 1), the motorist is 0.3 miles past the intersection (East of the intersection) with Route 1, and Officer Egbert trains his radar gun on the car, which tells him that the rate of change of the distance between his

squad car and the motorist is exactly 0. How fast is the motorist driving? *(10 points)*



Solution: [Similar to 11 and 12 on the review sheet, or 2 on the 2005 exam, or 8 on the 2008 exam] Set x to be the distance from the motorist to the intersection of the two highways, and y to be the distance from Officer Egbert to the intersection. Finally, set s to be the distance between the two cars. We are told that $dy/dt = -60$ when $x = 0.3$ and $y = 0.4$, and $ds/dt = 0$ at that time. We want to know dx/dt at that time. Since

$$x^2 + y^2 = s^2,$$

if we differentiate both sides,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}.$$

If you plug in $ds/dt = 0$ at the "when", you don't even need to find s, so plug that in, along with $x = 0.3$, $y = 0.4$, and $\frac{dy}{dt} = -60$,

$$\begin{aligned}
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2s\frac{ds}{dt} \\
2\,(0.3)\,\frac{dx}{dt} + 2\,(0.4)\,(-60) &= 2s \cdot 0, \text{ so} \\
2\,(0.3)\,\frac{dx}{dt} &= 2\,(0.4)\,(60)\,, \text{ or} \\
\frac{dx}{dt} &= \frac{2\,(0.4)\,(60)}{2\,(0.3)} \\
&= \frac{4\,(60)}{3} \\
&= 80,
\end{aligned}$$

so the motorist was going 80mph. Note that, as long as time was considered in terms of hours, and distances in miles, we automatically get the units being miles/hour.

(6)
(a) State the Mean Value Theorem. *(5 points)*
Solution: [26 on review sheet] If $f(x)$ is a continuous function on $[a, b]$ and is differentiable on (a, b), then there is a point $c \in (a, b)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

(b) Use the Mean Value Theorem to show that the function

$$f(x) := x^3 + 4x - 5$$

only has a root at $x = 1$, and nowhere else. *(5 points)*
Solution: [27 on review sheet, and 8 on 2005 exam, and 6 on the 2008 exam.] If there were two roots of that function, two points a and b at which $f(a) = f(b) = 0$, then by MVT there is a point c between a and b at which $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$. On the other hand, $f'(x) = 3x^2 + 4$, which is always at least 4, so is never 0. So such a c can't exist, which means that the assumption that there were two roots is false. On the other hand, $f(1) = 0$, so it does indeed have only that one root.

(7) Use linear approximation to find an approximate value of $(25)^{1/3}$, the cube root of 25. *(10 points)*
Solution: [Similar to 16, 17, and 19 on review sheet, and 3 on the 1996 exam, and 5 on the 2008 exam] You take the function $f(x) = x^{1/3}$, and $a = 27$ (because you can find the cube root there, and it is close to 25). Then, by linear approximation, noting that $f'(x) = \frac{1}{3x^{2/3}}$,

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x - a) \\ 25^{1/3} &\approx 27^{1/3} + \frac{1}{3 \cdot 27^{2/3}}(25 - 27) \\ &= 3 + \frac{1}{3 \cdot 9}(-2) \\ &= 3 - \frac{2}{27} \\ &= \frac{79}{27}. \end{aligned}$$

(8) Let $f(x) = \dfrac{x^2 - 1}{x^2 + 3}$, then *(20 points)* [Similar to 33, 34, 36, and all parts of 39 on the review sheet, plus 8 on the 2005 exam, 9 on the 1996 exam, and 10 on the 2008 exam]
(a) Find the domain of $f(x)$.
Solution: The domain is all real numbers. Note that the denominator is never 0.
(b) Find all x- and y- intercepts.
Solution: The y-intercept is $(0, f(0)) = (0, -1/3)$. The x-intercepts are $(x, 0)$ for those x where the numerator is 0, so $(1, 0)$ and $(-1, 0)$.
(c) Find any horizontal or vertical asymptotes.

Solution: There is no vertical asymptote, since the denominator never is 0. There is a horizontal asymptote, since

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+3} = 1,$$

and that asymptote is approached by the graph on both ends. For any x, though, the numerator is less than the denominator, so the graph approaches the horizontal asymptote from below on both ends.

Continuing with problem 8. Recall that $f(x) = \dfrac{x^2-1}{x^2+3}$

(d) Find where the curve is increasing and decreasing, and find any critical points.

Solution:

$$\begin{aligned} f'(x) &= \frac{2x(x^2+3) - 2x(x^2-1)}{(x^2+3)^2} \\ &= \frac{8x}{(x^2+3)^2}, \end{aligned}$$

so the curve is increasing ($f'(x) > 0$) when $x > 0$ and is decreasing ($f'(x) < 0$) when $x < 0$. Technically, you should say that f is increasing when $x \geq 0$ and decreasing when $x \leq 0$, but either way is OK. There is only one critical point, at $x = 0$, with value $f(0) = -1/3$, and that is a local minimum, by the first derivative test. You can also see that it is a local minimum from the second derivative test, which you will see in the next part. $x = 0$ is also an absolute (or global) minimum, since there is only one critical point which is a local minimum, hence an absolute minimum. You weren't asked that, though, only to find the critical point.

(e) Find on what regions the curve is concave up, and where it is concave down, and find any points of inflection.

Solution:

$$\begin{aligned} f''(x) &= \left(\frac{8x}{(x^2+3)^2}\right)' \\ &= \frac{8(x^2+3)^2 - \big(2(x^2+3)2x\big)\,8x}{(x^2+3)^4} \\ &= \frac{8(x^2+3) - (2\cdot 2x)\,8x}{(x^2+3)^3} \\ &= \frac{24-24x^2}{(x^2+3)^3} \\ &= \frac{24(1-x^2)}{(x^2+3)^3}, \end{aligned}$$

so the inflection points are at $x = \pm 1$. For $x > 1$, $f''(x) < 0$, and for $x < -1$, $f''(x) < 0$, but for $-1 < x < 1$, $f''(x) > 0$, so the curve is concave up on $(-1, 1)$ and concave down on $(-\infty, -1) \cup (1, \infty)$, with inflection points at $(1, 0)$ and $(-1, 0)$ (You already found the y-values since they are the x-intercepts).

Continuing with problem 8. Recall that $f(x) = \dfrac{x^2-1}{x^2+3}$

(f) Then, sketch the curve, showing each of these features.

Solution: Here is the sketch that Maple produces: