MATH 21, FALL, 2009, EXAM # 1 SOLUTIONS

(1) Find the indicated limits: Show the steps involved. *(5 points/part)*

(a) $\lim_{x\to 2} \dfrac{x^2-3x+2}{x^2+x-6} =$

Solution: Since both the numerator and denominator go to 0 as $x \to 2$, you should know that $(x-2)$ is a factor of both.

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6} &= \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)(x+3)} \\
&= \lim_{x\to 2} \frac{(x-1)}{(x+3)} \\
&= \frac{(2-1)}{(2+3)} \\
&= \frac{1}{5}.
\end{aligned}$$

(b) $\lim_{x\to\infty} \sqrt{x^2+2x-1} - x =$

Solution: Conjugate:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+2x-1} - x &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x-1} - x\right)\left(\sqrt{x^2+2x-1} + x\right)}{\left(\sqrt{x^2+2x-1} + x\right)} \\
&= \lim_{x\to\infty} \frac{\left(x^2+2x-1\right) - x^2}{\left(\sqrt{x^2+2x-1} + x\right)} \\
&= \lim_{x\to\infty} \frac{2x-1}{\left(\sqrt{x^2+2x-1} + x\right)} \\
&= \lim_{x\to\infty} \frac{(2x-1)\,\frac{1}{x}}{\left(\sqrt{x^2+2x-1} + x\right)\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{\sqrt{\frac{x^2+2x-1}{x^2}} + 1} \\
&= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{\sqrt{1+\frac{2}{x}-\frac{1}{x^2}} + 1} \\
&= \frac{2}{1+1} = 1.
\end{aligned}$$

(c) $\lim_{t\to 0} \dfrac{\sqrt{2-t}-\sqrt{2}}{t} =$

Solution: Again, conjugate:

$$\begin{aligned}
\lim_{t\to 0}\frac{\sqrt{2-t}-\sqrt{2}}{t} &= \lim_{t\to 0}\frac{\left(\sqrt{2-t}-\sqrt{2}\right)\left(\sqrt{2-t}+\sqrt{2}\right)}{t\left(\sqrt{2-t}+\sqrt{2}\right)}\\
&= \lim_{t\to 0}\frac{(2-t)-2}{t\left(\sqrt{2-t}+\sqrt{2}\right)}\\
&= \lim_{t\to 0}\frac{-t}{t\left(\sqrt{2-t}+\sqrt{2}\right)}\\
&= \lim_{t\to 0}\frac{-1}{\left(\sqrt{2-t}+\sqrt{2}\right)}\\
&= \frac{-1}{2\sqrt{2}}
\end{aligned}$$

(d) $\displaystyle\lim_{x\to 0}\frac{\tan(4x)}{\tan(3x)} =$

Solution:

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan(4x)}{\tan(3x)} &= \lim_{x\to 0}\frac{\tan(4x)}{}\frac{}{\tan(3x)}\\
&= \lim_{x\to 0}\frac{\sin(4x)}{4x}\frac{1}{\cos(4x)}\frac{\sin(3x)}{3x}\frac{1}{\cos(3x)}\frac{4x}{3x}\\
&= \lim_{t\to 0}\frac{\sin(t)}{t}\lim_{t\to 0}\frac{1}{\cos(t)}\lim_{s\to 0}\frac{\sin(s)}{s}\lim_{s\to 0}\frac{1}{\cos(s)}\frac{4}{3},\ \text{if } t=4x \text{ and } s=3x\\
&= (1)\cdot(1)\cdot(1)\cdot(1)\,\frac{4}{3}\\
&= \frac{4}{3}.
\end{aligned}$$

No, you are not supposed to use l'Hôpital's rule.

(2) Find the following derivatives, using the rules we have discussed in class. *(5 points/part)*

(a) $\left((x^2+2x-3)(x^3-5)\right)' =$

Solution:

$$\begin{aligned}
\left((x^2+2x-3)(x^3-5)\right)' &= (x^2+2x-3)'(x^3-5)+(x^2+2x-3)(x^3-5)'\\
&= (2x+2)(x^3-5)+(x^2+2x-3)3x^2.
\end{aligned}$$

Stop here, don't simplify.

(b) $\left(\dfrac{2x+3}{x^2-1}\right)' =$

Solution:

$$\begin{aligned}
\left(\frac{2x+3}{x^2-1}\right)' &= \frac{(2x+3)'\left(x^2-1\right)-\left(x^2-1\right)'(2x+3)}{\left(x^2-1\right)^2}\\
&= \frac{(2)\left(x^2-1\right)-(2x)(2x+3)}{\left(x^2-1\right)^2}\\
&= \frac{-2x^2-6x-2}{\left(x^2-1\right)^2},
\end{aligned}$$

although you should stop with that second line.

(c) $\left(\sin(x)\left(x^3+4x\right)\right)' =$

Solution:

$$\begin{aligned}\left(\sin(x)\left(x^3+4x\right)\right)' &= (\sin(x))'\left(x^3+4x\right)+\sin(x)\left(x^3+4x\right)' \\ &= \cos(x)\left(x^3+4x\right)+\sin(x)\left(3x^2+4\right).\end{aligned}$$

(d) If $f(x) = e^{2x+3}$, then $f'(x) =$

Solution:

$$f'(x) = 2e^{2x+3}.$$

(3) For the function $f(x) = \frac{2x+3}{x^2-1}$, find the vertical and horizontal asymptotes, and use those to make a rough sketch of the graph of $y = f(x)$. *(10 points)*

Solution: The vertical asymptotes occur when the denominator is 0, that is, when the limit of the function is infinite. In this case, that is at $x = 1$ and $x = -1$. The horizontal asymptote corresponds to limits as x goes to infinity, here the horizontal asymptote is the x-axis, $y = 0$. But, you can learn more by looking at one-sided limits. First, let's look at the two sides as $x \to 1$.

$$\lim_{x\to 1^+}\frac{2x+3}{x^2-1} = \lim_{x\to 1^+}\frac{2x+3}{(x+1)(x-1)},$$

so, as $x \to 1$, but $x > 1$, both terms in the denominator will be positive, as will the numerator. Of course, $(x-1)$ will be nearly 0, but it will be a little bit larger than 0, so the fraction is positive, and large. In the limit, then

$$\lim_{x\to 1^+}\frac{2x+3}{x^2-1} = +\infty.$$

Similarly (but now $(x-1) < 0$),

$$\lim_{x\to 1^-}\frac{2x+3}{x^2-1} = \lim_{x\to 1^-}\frac{2x+3}{(x+1)(x-1)}, = -\infty.$$

In the same way,

$$\lim_{x\to -1^+}\frac{2x+3}{x^2-1} = \lim_{x\to -1^+}\frac{2x+3}{(x+1)(x-1)}, = -\infty,$$

because $(x-1) < 0$, but $(x+1) > 0$ (small, but positive) and $2x+3 > 0$ as well, and

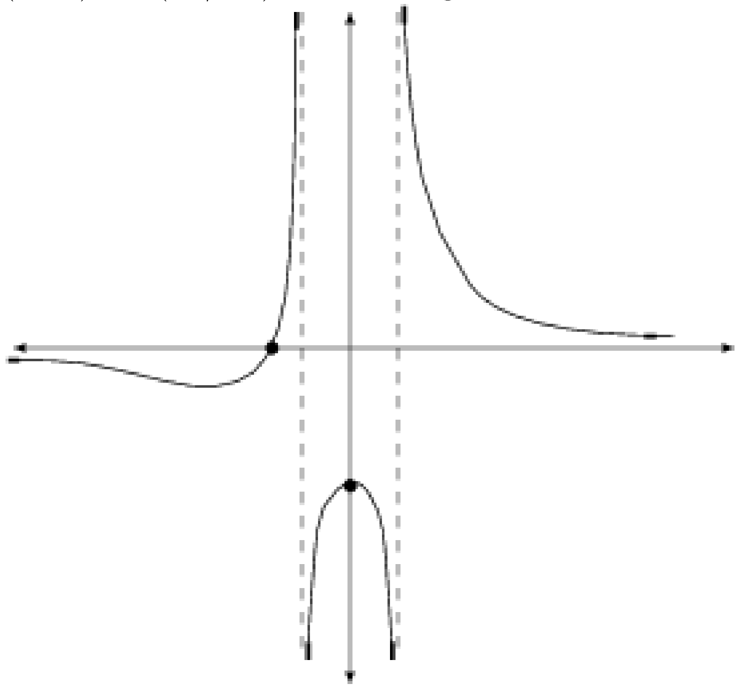
$$\lim_{x\to -1^-}\frac{2x+3}{x^2-1} = \lim_{x\to -1^-}\frac{2x+3}{(x+1)(x-1)}, = +\infty.$$

For the horizontal asymptote,

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{2x+3}{x^2-1} &= \lim_{x\to\pm\infty}\frac{(2x+3)\frac{1}{x}}{(x^2-1)\frac{1}{x}} \\ &= \lim_{x\to\pm\infty}\frac{2+\frac{3}{x}}{x-\frac{1}{x}} \\ &= 0,\end{aligned}$$

since the denominator is still going to ∞ even though, after dividing out, the numerator will go to 2. The numerator, for x large (or negative and large in magnitude) will head to 2, but the denominator will go to $\pm\infty$, $+\infty$ when $x > 0$ and $-\infty$ when $x < 0$, so the fraction will be positive, though small, for large $x > 0$, and negative for

$x < 0$ and x large in magnitude.
These limits tell you the ends of the graph. You can also note that $f(0) = -3$ and $f(x) = 0$ only when $x = -3/2$, which tells you where the graph crosses the axes, $(0, -3)$ and $(-3/2, 0)$. Here is the graph.



(4) Show, using and ϵ-δ argument, that $\lim_{x\to 2} x^2 + x = 6$. *(15 points)*

Solution: If you look first at scratch work, you need to make $\left|(x^2+x)-6\right|$ small just by taking x near 2. But

$$\begin{aligned} \left|(x^2+x)-6\right| &= \left|x^2+x-6\right| \\ &= |(x+3)(x-2)| \\ &= |x+3|\,|x-2|\,. \end{aligned}$$

Now, $|x-2|$ can be made as small as we want, but we need to also control $|x+3|$. So, we start with an initial estimate on x — remember, all we can do is control how close to 2 x is. So, initially let's say $|x-2| < 1$, which puts x in the range: $-1 < x-2 < 1$ or $1 < x < 3$. Since x is in that range, $4 < x+3 < 6$, so, no matter what, $|x+3| < 6$. Now, look back up at the work we did for $\left|(x^2+x)-6\right|$. We need to keep x in the range $1 < x < 3$ so this estimate will work, and we need $|x-2|$ small enough so that, even when multiplied by as big as $|x+3|$ can be (which is 6), the value of $\left|(x^2+x)-6\right|$ is less than ϵ. So, $|x-2|$ has to be < 1 *and also* $< \epsilon/6$.
Now, to the proof.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \min\left\{1, \frac{\epsilon}{6}\right\}$. Then, whenever $0 < |x-2| < \delta$,

$$\begin{aligned} \left|(x^2+x)-6\right| &= \left|x^2+x-6\right| \\ &= |(x+3)(x-2)| \\ &= |x+3|\,|x-2| \\ &< 6\,|x-2| \\ &< 6\frac{\epsilon}{6} = \epsilon. \end{aligned}$$

□

(5) Let

$$f(x) := \begin{cases} x^2, & \text{if } x \le 2 \\ mx+6, & \text{if } x > 2 \end{cases}.$$

Find the values of m for which the function f will be continuous at $x = 2$. *(10 points)*

Solution: The function will be continuous when

$$\lim_{x\to 2} f(x) = f(2) = 4.$$

At all other points the function is certainly continuous. But, in order for this limit to exist, the two one-sided limits have to exist, and be equal. Those one-sided limits are:

$$\begin{aligned} \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} mx+6 \\ &= 2m+6, \end{aligned}$$

because the function has values $mx+6$ for $x > 2$, and

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} x^2 \\ &= 4, \end{aligned}$$

since $f(x) = x^2$ for $x > 2$. In order for these two one-sided limits to agree, we need

$$2m+6 = 4$$

or $m = -1$.

(6)

(a) State the definition of the derivative of a function $f(x)$ (as a limit): *(5 points)*
$f'(x) =$

Solution: The *definition* of the derivative of a function $f(x)$, as a limit, is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

(b) Use this definition to determine $f'(x)$, for the function $f(x) = \dfrac{1}{x^2}$. *(10 points)*

Solution:

$$\begin{aligned}
\left(\frac{1}{x^2}\right)' &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ with } f(x)=\frac{1}{x^2}, \text{ so}\\
&= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\\
&= \lim_{h\to 0}\frac{\left(\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right)}{h}\\
&= \lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}\\
&= \lim_{h\to 0}\frac{-2xh-h^2}{hx^2(x+h)^2}\\
&= \lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2}\\
&= \frac{-2x}{x^4}\\
&= \frac{-2}{x^3}.
\end{aligned}$$

(7) Suppose that $f(x)$ and $g(x)$ are differentiable functions so that $f(1)=1$, $f'(1)=4$, $f(4)=-3$, $f'(4)=1$, $f(5)=3$, $f'(5)=2$, $g(1)=5$, $g'(1)=4$, $g(2)=1$, and $g'(2)=3$. Find $\frac{d}{dx}\left[f(g(x))\right]\big|_{x=1}$. *(10 points)*

Solution: By the chain rule,

$$\begin{aligned}
\frac{d}{dx}\left[f(g(x))\right]\big|_{x=1} &= f'(g(1))g'(1)\\
&= f'(5)\cdot 4\\
&= 2\cdot 4 = 8.
\end{aligned}$$