MATH 21, FALL, 2008, FINAL EXAM

(1) Find the indicated limits: Show the steps involved. *(5 points/part)*

(a) $\lim_{t\to 1} \frac{t^2-3t+2}{t^2-1}$

Solution::

$$\begin{aligned}\lim_{t\to 1} \frac{t^2-3t+2}{t^2-1} &= \lim_{t\to 1} \frac{(t-1)(t-2)}{(t-1)(t+1)} \\ &= \lim_{t\to 1} \frac{(t-2)}{(t+1)} \\ &= \frac{-1}{2}.\end{aligned}$$

(b) $\lim_{x\to 0^+} \sqrt{x}\ln x$

Solution:: This one uses l'Hôpital's Rule, ∞/∞ form

$$\begin{aligned}\lim_{x\to 0^+} \sqrt{x}\ln x &= \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{\sqrt{x}}\right)} \\ &= \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{\sqrt{x}}\right)} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(\frac{-1}{2x^{3/2}}\right)} \\ &= \lim_{x\to 0^+} \frac{-2x^{3/2}}{x} \\ &= \lim_{x\to 0^+} -2x^{1/2} \\ &= 0.\end{aligned}$$

(c) $\lim_{x\to\infty} x - \frac{x^2+1}{x+1}$

Solution:: This should be simplified first,

$$\begin{aligned}\lim_{x\to\infty} x - \frac{x^2+1}{x+1} &= \lim_{x\to\infty} \frac{x\,(x+1) - \left(x^2+1\right)}{x+1} \\ &= \lim_{x\to\infty} \frac{x-1}{x+1} \\ &= 1.\end{aligned}$$

(2) Find the indicated derivatives: Show the steps involved. Do not simplify. *(5 points/part)*

(a) $\left((x^2+2x-3)(x^3-5)\right)'$

Solution::

$$\begin{aligned}\left((x^2+2x-3)(x^3-5)\right)' &= (x^2+2x-3)'(x^3-5) + (x^2+2x-3)(x^3-5)' \\ &= (2x+2)(x^3-5) + (x^2+2x-3)(3x^2)\end{aligned}$$

(b) $\left(\dfrac{x^2+3x-2}{\sqrt{x+2}}\right)'$

Solution::

$$\begin{aligned}\left(\frac{x^2+3x-2}{\sqrt{x+1}}\right)' &= \left(\frac{\left(x^2+3x-2\right)'\sqrt{x+1}-\left(\sqrt{x+1}\right)'\left(x^2+3x-2\right)}{\left(\sqrt{x+1}\right)^2}\right)\\ &= \left(\frac{(2x+3)\sqrt{x+1}-\left(\frac{1}{2\sqrt{x+1}}\right)\left(x^2+3x-2\right)}{\left(\sqrt{x+1}\right)^2}\right)\\ &= \left(\frac{(2x+3)(x+1)-\frac{1}{2}\left(x^2+3x-2\right)}{(x+1)^{3/2}}\right).\end{aligned}$$

(c) $\left(\sin^3 x+\sin(x^3)\right)'$

Solution::

$$\left(\sin^3 x+\sin(x^3)\right)' = 3\sin^2(x)\cos(x)+3x^2\cos(x^3).$$

Remember that $\sin^3 x = (\sin(x))^3$.

(d) Assume that $y=f(x)$ satisfies the equation

$$y^5+3x^2y^2+5x^4=12.$$

Find dy/dx in terms of x and y.

Solution::

$$\begin{aligned}5y^4\frac{dy}{dx}+6xy^2+6x^2y\frac{dy}{dx}+20x^3 &= 0,\ \text{or}\\ \left(5y^4+6x^2y\right)\frac{dy}{dx} &= -6xy^2-20x^3,\ \text{so}\\ \frac{dy}{dx} &= -\frac{6xy^2+20x^3}{5y^4+6x^2y}.\end{aligned}$$

(e) $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^2}\frac{1}{t^3+1}\,dt\right)$

Solution:: If $F(x)=\int_1^x \frac{1}{t^3+1}\,dt$, then what we have to differentiate here is $F(x^2)$. We use the chain rule and the first part of the FTC to say that

$$\begin{aligned}\frac{d}{dx}\left(\int_1^{x^2}\frac{1}{t^3+1}\,dt\right) &= F'(x^2)2x\\ &= \frac{1}{(x^2)^3+1}2x\\ &= \frac{2x}{x^6+1}.\end{aligned}$$

(3) Evaluate the following integrals. *(5 points/part)*

(a) $\displaystyle\int (x^3+3x^2-4x+2)\,dx$

Solution::

$$\int (x^3+3x^2-4x+2)\,dx = \frac{1}{4}x^4+x^3-2x^2+2x+c.$$

(b) $\displaystyle\int_0^{\pi/2}\sin(x)\cos(x)\,dx$

Solution: Substitute $u = \sin(x)$. Then $du = \cos(x)dx$, when $x = \pi/2$, $u = 1$, and when $x = 0$, $u = 0$. So

$$\begin{aligned}\int_0^{\pi/2} \sin(x)\cos(x)\,dx &= \int_0^1 udu \\ &= \frac{1}{2}u^2\Big|_0^1 \\ &= \frac{1}{2}.\end{aligned}$$

Notice that I did not substitute back for u in terms of x, since I had changed the limits of integration. It is often better to do it this way, and never worse.
You could also do this with $u = \cos(x)$ and $du = -\sin(x)dx$. Except for the negative signs, it works the same way.

(c) $\displaystyle\int_{-1}^{1} \frac{x}{1+x^2}\,dx$

Solution: Substitute $u = 1 + x^2$. Then $du = 2xdx$, when $x = 1$, $u = 2$ and when $x = -1$, $u = 2$ Hmm. So,

$$\begin{aligned}\int_{-1}^{1} \frac{x}{1+x^2}\,dx &= \frac{1}{2}\int_2^2 \frac{du}{u} \\ &= 0.\end{aligned}$$

Here you don't even have to do the integration, since the limits are from 2 to 2. There is no area under that curve, because the interval has length 0. You could to the integration, but it would give the same result. This is one where it is really easier than it would be if you took the trouble to substitute back in terms of x. Alternately, you can see that the original integral is 0 because the integrand is odd, and the original integral is over an interval with 0 at the center.

(d) $\displaystyle\int x\sqrt{x-1}\,dx$

Solution:: This is a tricky u-substitution, $u = x - 1$. Then, of course $du = dx$, but you also substitute for x as $x = u + 1$, so that

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt{u}du \\ &= \int u^{3/2} + \sqrt{u}du \\ &= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c \\ &= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c\end{aligned}$$

(e) $\displaystyle\int_3^{11} \frac{dx}{\sqrt{2x+3}}$

Solution: Substitute $u = 2x + 3$. Then $du = 2dx$, when $x = 3$, $u = 9$ and when $x = 11$, $u = 25$, so

$$\begin{aligned}\int_3^{11} \frac{dx}{\sqrt{2x+3}} &= \frac{1}{2}\int_9^{25} \frac{du}{\sqrt{u}} \\ &= \sqrt{u}\Big|_9^{25} \\ &= 5 - 3 \\ &= 2.\end{aligned}$$

(4) (a) State the definition of the derivative of a function $f(x)$ (as a limit): $f'(x) =$ *(5 points)*

Solution:: The *definition* of the derivative of a function $f(x)$, as a limit, is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

(b) Using only the definition of the derivative, find the derivative function $f'(x)$ for the function $f(x) = \sqrt{x}$. *(5 points)*

Solution:: Now, to apply to $f(x) = \sqrt{x}$,

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{(x+h) - (x)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}.\end{aligned}$$

(5) Find $\lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)$. [Hint: This limit is an integral. Figure out what function is being integrated, and over what interval.] *(10 points)*

Solution:: Use the $\frac{\pi}{n}$ factor as the width of each subinterval. Then, since there are n subintervals (n terms in the sum), the interval stretches from $x = 0$ to $x = \pi$. One trick with these problems is that you can always set them up with the left-edge of the interval taken to be 0. Since the width of each subinterval is $\frac{\pi}{n}$, and we start at 0, then $x_i = i\frac{\pi}{n}$. So, since a Riemann sum will be $\sum f(x_i)\Delta x$, we see that $f(x) = \cos(x)$, and

$$\begin{aligned}\lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right) &= \int_0^{\pi} \cos(x)dx \\ &= \sin(x)\Big|_0^{\pi} \\ &= 0.\end{aligned}$$

(6) A boater has his boat tied to the pier by a long rope. He feeds the pier-end of the rope through a winch and pulls his boat in towards the pier. The tide is out, so the bow of the boat is 6 feet below the level of the pier. If the winch is pulling in the line at a constant rate of 2 feet per second, how fast is the boat moving towards the pier when there is 10 feet of rope between the boat and the winch? *(10 points)*

Solution:: Take x to be the horizontal distance between the boat and the pier, and s to be the length of rope that is between the wench and the boat. Then

$$x^2 + 6^2 = s^2$$

is the equation relating those variables. We **know** $\frac{ds}{dt} = -2$ (the rope is being pulled in by the winch), and we **want** $\frac{dx}{dt}$ **when** $s = 10$. Differentiate the equation:

$$2x\frac{dx}{dt} + 0 = 2s\frac{ds}{dt},$$

then plug in for what we know and the when:

$$2x\frac{dx}{dt} + 0 = 2(10)(-2).$$

We also use the equation to find x at this time,

$$x^2 + 6^2 = 10^2$$

which is a 3-4-5 triangle, so $x = 8$. Plug that in, and we can solve for $\frac{dx}{dt}$:

$$\begin{aligned} 2(8)\frac{dx}{dt} + 0 &= 2(10)(-2) \\ \frac{dx}{dt} &= -\frac{5}{2}, \end{aligned}$$

or the boat is moving towards the pier at 5/2 feet/sec.

(7) Take a 12 inch by 12 inch rectangle of tin, cut equal squares out of each corner and bend up the sides, then solder the seams on the edges to make a rectangular box with no top. How large a square should you cut out of each corner to maximize the volume of the resulting box? *(10 points)*

`tin-box.ps`

Solution:: If the cut-out corners are all $x \times x$, then the remaining base is a square that is $(12-2x) \times (12-2x)$. Since x is a physical measurement, and you can't cut out more than 1/4 of the whole sheet from each corner, or else you are cutting air instead of the next corner, $0 \le x \le 6$. The volume of the box with the $x \times x$ corners cut out is

$$V = (12 - 2x)^2 x,$$

and there is no constraint. So, differentiate:

$$\begin{aligned} V' &= (2(12-2x)(-2))\,x + (12-2x)^2 \\ &= (12-2x)(-4x + (12-2x)) \\ &= (12-2x)(-6x+12), \end{aligned}$$

which is zero only when $x = 6$ or $x = 2$. Now we just plug in at all the possible points, $x = 0,\ 2$, or 6, the critical points and endpoints ($x = 6$ counts as both).

$$\begin{aligned} V(0) &= 0, \\ V(2) &= 128, \\ V(6) &= 0. \end{aligned}$$

Clearly $x = 2$ gives the maximum. The question, though, was to find the size of the squares cut out, which would be 2×2.

(8) Find the linear approximation of $f(x) = \sqrt{x+1}$ for x near 0. Your answer should be in the form $g(x) = Ax + B$. *(10 points).*

Solution:: Using linear approximation, $f(x) \approx f(a) + f'(a)(x - a)$, with $f(x) = \sqrt{x+1}$ and $a = 0$ gives, since $f(0) = 1$ and $f'(x) = \frac{1}{2\sqrt{x+1}}$, so $f'(0) = \frac{1}{2}$,

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x - a) \\ &= 1 + \frac{1}{2}(x - 0) \\ &= \frac{1}{2}x + 1. \end{aligned}$$

(9) 100 grams of pure Doublemintium (D_m), a new radioactive substance, was found under the seats of Neville II. After 1 day there was 90 grams left, the rest having decomposed into micronite and tar. Presuming that D_m, like any radioactive substance, decomposes at a rate proportional to the amount present *(10 points)*:

(a) Find a formula for the amount present at time t.

Solution: Since the amount of D_m present in the sample decays away at a rate proportional to the amount present, if $y(t)$ is the amount of D_m present at time t, then $y'(t) = ky(t)$, which means that $y(t) = Ae^{kt}$ as with a population problem. since 100g was initially found, $A = 100$, and so $y = 100e^{kt}$. But also,

$$\begin{aligned} 90 &= y(1) \\ &= 100e^{k}, \end{aligned}$$

and so $\frac{9}{10} = e^k$, or $k = \ln(9/10)$. So

$$y(t) = 100e^{\ln(9/10)t}.$$

(b) When will half of the original amount be present?

Solution:: Half of the original amount will be present when

$$\begin{aligned} 50 &= y(t) \\ &= 100e^{\ln(9/10)t}, \end{aligned}$$

so

$$\begin{aligned} \frac{1}{2} &= e^{\ln(9/10)t}, \\ \ln(1/2) &= \ln(9/10)t, \end{aligned}$$

or

$$t = \frac{\ln(1/2)}{\ln(9/10)}.$$

(10) *(5 points/part)*

(a) Approximate the integral $\int_0^4 \left(x^2 - 2x - 3\right)\, dx$ by using a Riemann sum with 4 subintervals, using right-hand endpoints.

Solution:: Since we break up $[0,4]$ into 4 subintervals, then $x_i = i$, $i = 1,2,3,4$. So the sum will be:

$$\begin{aligned}\sum_{i=1}^{4} f(x_i)\Delta x_i &= f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 \\ &= (1-2-3)+(4-4-3)+(9-6-3)+(16-8-3) \\ &= -4-3+0+5 \\ &= -2.\end{aligned}$$

(b) Approximate the integral $\int_0^4 \left(x^2-2x-3\right)\,dx$ by using a Riemann sum with an arbitrary number n of subintervals, using right-hand endpoints. Leave your answer as a sum.

Solution:: Here, $x_i = \frac{4}{n}i$ since we break $[0,4]$ up into n subintervals. $\Delta x_i = \frac{4}{n}$ and $f(x_i) = x_i^2 - 2x_i - 3$

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x_i &= \sum_{i=1}^{n}\left(\left(\frac{4}{n}i\right)^2 - 2\left(\frac{4}{n}i\right) - 3\right)\frac{4}{n} \\ &= \left(\frac{4}{n}\right)^3\sum_{i=1}^{n} i^2 - 2\left(\frac{4}{n}\right)^2\sum_{i=1}^{n} i - 3\left(\frac{4}{n}\right)\sum_{i=1}^{n} 1 \\ &= \left(\frac{4}{n}\right)^3\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{4}{n}\right)^2\frac{n(n-1)}{2} - 3\left(\frac{4}{n}\right)n,\end{aligned}$$

but I suppose that that last line wasn't needed to satisfy the requirements of the problem.

(c) State the second part of the Fundamental Theorem of Calculus, and use that to find the exact value of $\int_0^4 \left(x^2-2x-3\right)\,dx$.

Solution:: FTC-part 2: If $f(x)$ is a continuous function on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$.
Then,

$$\begin{aligned}\int_0^4 \left(x^2-2x-3\right)\,dx &= \left.\frac{1}{3}x^3 - x^2 - 3x\right|_0^4 \\ &= \left(\frac{1}{3}(4)^3 - (4)^2 - 3(4)\right) - \left(\frac{1}{3}(0)^3 - (0)^2 - 3(0)\right) \\ &= \frac{64}{3} - 16 - 12 \\ &= -\frac{20}{3},\end{aligned}$$

where again that last line is more than you should bother with.

(11) Find the integral

$$\int_{-3}^{3}\left(1+\sqrt{9-x^2}\right)\,dx$$

by interpreting it in terms of areas. *(10 points)*

Solution:: The graph of $y = \sqrt{9-x^2}$ is a semi-circle, corresponding to $x^2+y^2=9$, so a semi-circle of radius 3. The 1+ means that that semi-circle is on top of the rectangle from -3 to +3, of height 1, so the area is

$$\begin{aligned}\int_{-3}^{3}\left(1+\sqrt{9-x^2}\right)\,dx &= 6+\frac{1}{2}\pi 3^2\\ &= 6+\frac{9}{2}\pi.\end{aligned}$$

(12) If a particle is moving along a line so that its acceleration $a(t) = t^2 + 3\cos(t)$, with velocity $v(0) = 3$ at time $t = 0$ and position $s(0) = 2$ at time $t = 0$, find $s(t)$. *(10 points)*

Solution: Since the acceleration is $a(t) = t^2 + 3\cos(t)$, the velocity $v(t)$ is

$$\begin{aligned}v(t) &= \int a(t)dt\\ &= \int t^2 + 3\cos(t)\,dt\\ &= \frac{1}{3}t^3 + 3\sin(t) + C,\end{aligned}$$

and since $v(0) = 3$,

$$\begin{aligned}3 &= v(0)\\ &= \frac{1}{3}\cdot 0^3 + 3\sin(0) + C\\ &= C,\end{aligned}$$

so $v(t) = \frac{1}{3}t^3 + 3\sin(t) + 3$. Similarly,

$$\begin{aligned}s(t) &= \int v(t)dt\\ &= \int \frac{1}{3}t^3 + 3\sin(t) + 3\,dt\\ &= \frac{1}{12}t^4 - 3\cos(t) + 3t + C,\end{aligned}$$

but here C is not just $s(0)$. You need to use the condition to solve for C,

$$\begin{aligned}2 &= s(0)\\ &= \frac{1}{12}\cdot 0^4 - 3\cos(0) + 3\cdot 0 + C\\ &= -3 + C\end{aligned}$$

so $C = 5$, and $s(t) = \frac{1}{12}t^4 - 3\cos(t) + 3t + 5$.

(13) (a) Let $g(x) = \int_0^x e^{t^2}\,dt$. State the conclusion of the first part of the Fundamental Theorem of Calculus for this function $g(x)$, and then find $g''(x)$ (note, two derivatives) and all intervals where the graph of $y = g(x)$ is concave up, and all intervals where the graph is concave down. *(15 points)*

Solution:: The conclusion from the first part of the FTC for this function is that g(x) is differentiable and $g'(x) = e^{x^2}$. Now, $g''(x) = 2xe^{x^2}$ by the chain rule. Since $g''(x) > 0$ for $x > 0$ and $g''(x) < 0$ for $x < 0$, the curve is concave up for $x > 0$ and is concave down for $x < 0$, and has an inflection point at $x = 0$.

(b) Where is the function $G(x) = \int_0^x \frac{(t+1)(t+2)}{t^4+1}dt$ increasing, and where is it decreasing? What is $G(0)$? Is $G(-1)$ positive or negative? *(10 points)*

Solution:: Since, by the FTC again, $G'(x) = \frac{(x+1)(x+2)}{x^4+1}$, $G'(x) > 0$ when $x > -1$ or $x < -2$, and $G'(x) < 0$ when $-2 < x < -1$. So, $G(x)$ is increasing on $(-\infty, -2]$ and on $[-1, \infty)$, and $G(x)$ is decreasing on $[-2, -1]$. The endpoints are included for both, since on those intervals the function is increasing, even up to the endpoints. $G(0) = \int_0^0 \frac{(t+1)(t+2)}{t^4+1} dt = 0$ because it's the integral from 0 to 0. $G(x)$ is increasing on $[-1, 0]$, and $G(0) = 0$, so $G(-1)$ has to be negative.

(14) Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$. *(15 points)*

Solution:: The region bounded by these curves has $y = x + 2$ as the "top" and $y = x^2$ as the bottom, and the $x's$ needed range from $x = -1$ to $x = 2$. Those are the intersections of the two curves, where $x^2 = x + 2$, or $x^2 - x - 2 = 0$. You can factor that quadratic as $(x - 2)(x + 1) = 0$, so the intersections are at $x = -1$ and $x = 2$. So, the area is

$$\begin{aligned}
\int_{-1}^{2} (x+2) - x^2 \, dx &= \left. \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \right|_{-1}^{2} \\
&= \left(\frac{1}{2}(2)^2 + 2(2) - \frac{1}{3}(2)^3 \right) - \left(\frac{1}{2}(-1)^2 + 2(-1) - \frac{1}{3}(-1)^3 \right) \\
&= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \\
&= \frac{9}{2}.
\end{aligned}$$

Of course, you don't need to take that final step.