MATH 21, FALL, 2008, FINAL EXAM REVIEW

Suggestions. Imagine that you are working the problems on an exam with no help at all. Remember, a problem on the exam won't tell you what section of the book or notes it comes from, so it's important to spend some time working problems in random order. Work without a calculator since no calculators are allowed.

For the exam, you will be required to show all work. No calculators will be allowed. If you encounter an expression which is hard to simplify, leave it as it is. Please remember, however, to evaluate trigonometric, inverse trigonometric, logarithmic, and exponential functions at the given value whenever possible (for example $\sin(\pi/2)$).

(1) Evaluate the following limits, if they exist. If the limit does not exist, explain why.

(a) $\displaystyle\lim_{x\to -1} \frac{x^2-1}{x+1}$

(b) $\displaystyle\lim_{x\to 1^-} \frac{x+2}{x-1}$

(c) $\displaystyle\lim_{x\to \infty} \frac{x^2+1}{2x^2-1}$

(d) $\displaystyle\lim_{x\to 2^+} \frac{1}{\ln(x-2)}$

(e) $\displaystyle\lim_{x\to 0^+} \sqrt{x}\ln x$

(f) $\displaystyle\lim_{x\to 0} \frac{\sin x}{3x}$

(g) $\displaystyle\lim_{x\to \infty} \frac{\sqrt{2x^2+5x+1}}{x}$

(h) $\displaystyle\lim_{x\to 2} \frac{|x-2|}{x-2}$

(i) $\displaystyle\lim_{x\to 1^-} \arcsin x$

(j) $\displaystyle\lim_{x\to \infty} \left(1+\frac{r}{x}\right)^x$

(k) $\displaystyle\lim_{h\to 0} \frac{4^h-1}{h}$

(l) $\displaystyle\lim_{x\to 0} \frac{\arctan x}{e^x-1}$

(m) $\displaystyle\lim_{x\to \infty} x-\frac{x^2+1}{x+1}$

(n) $\displaystyle\lim_{n\to \infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{6i}{n}\right)+1\right]$ This is an integral, given as a limit of sums. What integral?

(2) Evaluate the following

(a) $\displaystyle\frac{d}{dx}(e^{-2x}+3\ \ln|x|)\Big|_{x=-1/2}$

(b) $(\csc x/(1+x^2))'$

(c) $\displaystyle\frac{d}{dx}\ln|3x-2|$

(d) $\displaystyle\frac{d}{dx}\sin^2(\cos(1+x^2))$

(e) $\left.\dfrac{d}{dx}(\arctan(x))^2\right|_{x=-1}$

(f) $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^2} \sin^3(t)\,dt\right)$

(g) $(x^{x^2+x})'$

(3) Suppose that $f(x)$ and $g(x)$ are differentiable functions so that $f(1) = 2$, $f'(1) = 3$, $f(4) = 2$, $f'(4) = 3$, $f(5) = -3$, $f'(5) = 1$, $g(1) = 4$, $g'(1) = 5$, $g(2) = 1$, and $g'(2) = 3$. Find $\frac{d}{dx}\left[f(g(x))\right]|_{x=1}$.

(4) Suppose that $f(x)$ and $g(x)$ are differentiable functions so that $f(1) = 1$, $f'(1) = 4$, $f(5) = 3$, $f'(5) = 2$, $f(4) = -3$, $f'(4) = 1$, $g(1) = 5$, $g'(1) = 4$, $g(2) = 1$, and $g'(2) = 3$. Find $\frac{d}{dx}\left[f(g(x))\right]|_{x=1}$

(5) Assume that $y = f(x)$ satisfies the equation

$$y^5 + 3x^2y^2 + 5x^4 = 12.$$

Find dy/dx in terms of x and y.

(6) If $x^2 - 3xy + y^2 = 5$, find $\frac{dy}{dx}$ at the point (1,-1). *(10 points).*

(7) Find an equation of the tangent line to the graph of $x^3 - xy - y^2 + 5 = 0$ at $(1, 2)$.

(8) Show, using the definition of the inverse tangent and implicit differentiation, that

$$\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}.$$

(9) The population of certain bacteria grows at a rate proportional to its size. It increases by 60% after 3 days. How long does it take for the population to double? (12 points)

(10) A new radioactive substance, Doublemintium (D_m), has been found sticking to the undersides of the seats in Packard Auditorium. 70 grams of pure D_m was collected initially. 5 days later only 60 grams of the stuff was still D_m; the rest had decayed into lead and tar. Assuming that the rate of decay of D_m is proportional to the amount present, how much will there be after 20 days?

(11) I bought a cup of coffee at McBurger's. It was far too hot to drink, 90^o Celsius. After 10 minutes, the coffee is at 80^o. The air in McBurger's is kept at an air-conditioned constant of 25^o. How long will I have to wait until the coffee is 70^o and thus cool enough to drink?

(12) A spherical snowball melts in such a way that its volume decreases at the rate of 2 cubic centimeters per minute. At what rate is the radius decreasing when the volume is 400 cubic centimeters?

(13) Egbert is flying his kite. It is 75 feet off the ground, moving horizontally away from Egbert at a rate of 5 feet per second. How fast is Egbert letting out the string when the kite is 100 feet (horizontally) downwind of him?

(14) A 20 foot ladder is leaning against a wall. A painter stands on the top of the ladder, minding his own business. Some fool comes by and ties his dog to the base of the ladder, a cat comes along, and the dog chases after the cat, dragging the base of the ladder with him at a rate of 2 feet per second directly away from the wall. How fast is the painter falling when he is 12 feet from the ground?

(15) Egbert is drinking a daiquiri (non-alcoholic) out of a glass that is a cone with a radius at the top of 3 inches and a height of 5 inches. He drinks his daiquiri at a constant rate of 2 cubic inches per second (through a straw). How fast is the top of the daiquiri falling when it is 4 inches above the bottom of the glass?

(16) A man, walking at night, is walking directly towards a streetlight. The light is 10 feet off the ground, and the man is 6 feet tall and walking at 4 feet per second. When the man is 8 feet from the streetlight, how fast is the length of his shadow changing?

(17) The side of a tall building is illuminated by a floodlight mounted on the lawn in front of the building. The floodlight is at ground level, and is 20 feet from the building. A man, 6

feet tall, is walking towards the floodlight at 4 feet/sec. How fast is his shadow on the wall growing when he is 10 feet from the floodlight?

(18) Use differentials to approximate

$$\sqrt{24}.$$

(19) Given that the function f satisfies $1 \le f(x) \le x^2 + 2x + 2$ for all x, find $\lim_{x \to -1} f(x)$.

(20) Does the equation $x^3 - x^2 = 3$ have a solution in the interval $1 < x < 2$? Justify your answer.

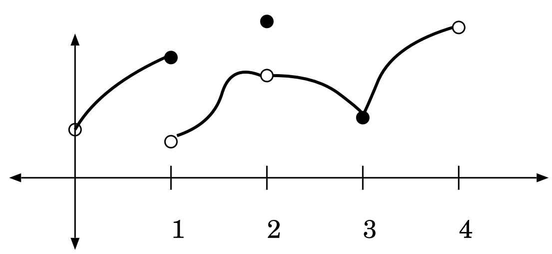
(21) Let $f(x) = x^2 + 2x$. Find $f'(1)$ using only the limit definition of the derivative.

(22) Below is the graph of a function f defined on the open interval $0 < x < 4$. Find those values of c with $0 < c < 4$ for which

(a) $\lim_{x \to c} f(x)$ does *not* exist.

(b) f is *not* continuous at c.

(c) f is *not* differentiable at c.

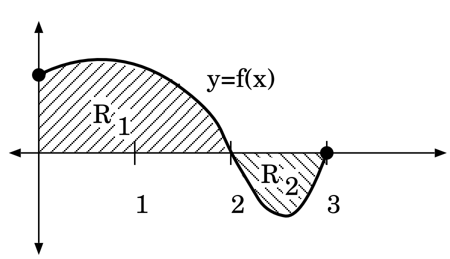


(23) Find an equation for the tangent line to the graph of $x^3 - xy - y^2 + 5 = 0$ at $(1, 2)$.

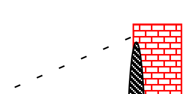
(24) Estimate $\sqrt{15.5}$ using differentials.

(25) A spherical balloon is being inflated with air at the rate of 2 cubic inches per second. At what rate is the radius expanding when when the volume is 20π cubic inches?

(26) The graph of a continuous function f determines two regions R_1 and R_2 as indicated below. The region R_1 has area $3/2$ and the region R_2 has area $5/8$. Find $\int_0^3 f(x)\,dx$.

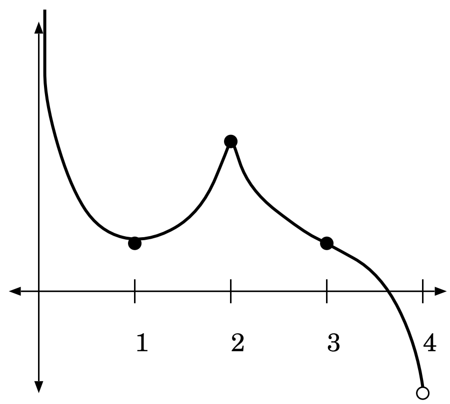


(27) Evaluate

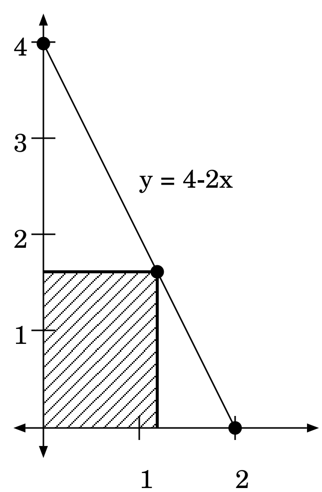


(a) $\int \left(2+\frac{1}{\sqrt{x}}+2e^{-3x}\right)\,dx$
(b) $\int \left(\frac{2}{1+x^2}-3\sin x\right)\,dx$
(c) $\int x\sqrt{1+x^2}\,dx$
(d) $\int_1^3 \left(x^2-\frac{1}{x}\right)\,dx$

(28) Below is the graph of a function f defined on the open interval $0 < x < 4$. Find
(a) The critical numbers
(b) The local extrema
(c) The absolute extrema
(d) Where the graph is concave up and where concave down.
(e) The inflection points.



(29) Find the largest possible area of a rectangle inscribed in the triangle bounded by $x = 0, y = 0$, and $y = 4 - 2x$ as indicated below. Please verify that your answer is really the maximum.

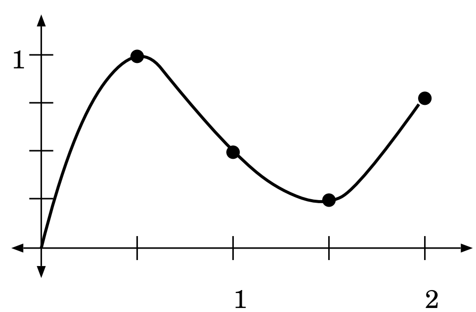


(30) Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.
(31) Sketch the graph of the function $f(x) = \frac{1}{4}x^4 - x^3$. Include (with labels) all local extreme points, all absolute extreme points, all inflection points, and all intercepts.
(32) Find $f'(x)$ for
(a) $f(x) = 4x^3 - x + 1$

(b) $f(x) = \dfrac{1}{x^2 + \sqrt{x}}$
(c) $f(x) = \dfrac{x+1}{x-1}$
(d) $f(x) = \dfrac{2}{x^5}$
(e) $f(x) = \ln|x| - 3e^{2x}$
(f) $f(x) = \tan^{-1} x \sin^{-1} x$
(g) $f(x) = \cos(x^2)$
(h) $f(x) = x^x$
(i) $f(x) = \displaystyle\int_5^x \sqrt{1+t^3}\, dt$

(33) Evaluate the following.
(a) $\displaystyle\int (x^3 + x^{1/3})\, dx$
(b) $\displaystyle\int \sinh(x)\cosh(x)\, dx$
(c) $\displaystyle\int x^2 e^{x^3}\, dx$
(d) $\displaystyle\int \frac{1+x}{1+x^2}\, dx$
(e) $\displaystyle\int x\sqrt{x-1}\, dx$

(34) Use four rectangles and left endpoints to approximate the area of the region pictured below.



(35) Determine if the function

$$f(x) := \begin{cases} x^2 - 4x + 2, & \text{if } x \geq 3 \\ \frac{\sin(x-3)}{x-3}, & \text{if } x < 3 \end{cases}$$

is continuous at 3.

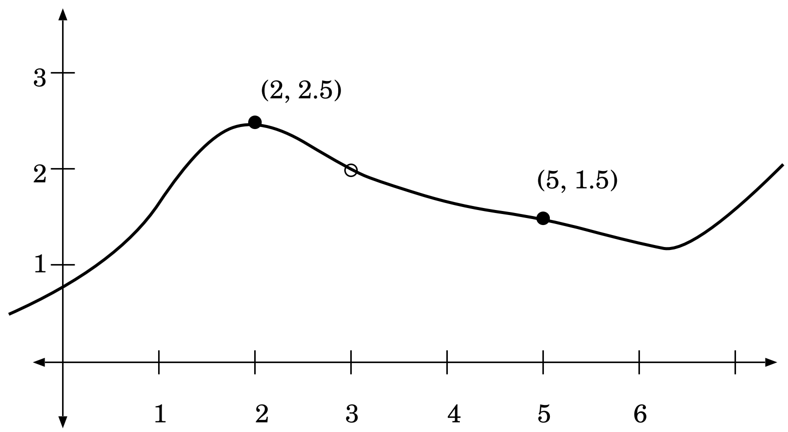
(36) Let $f(x) = \dfrac{\ln x}{x}$
(a) Identify the horizontal asymptotes of this function.
(b) Find an equation of the line tangent to the graph of this function at $x = 2$.
(c) Find the area of the region under the graph of this function over the interval $[1, e]$.

(37) A box with a square base and open top must have a volume of 32,000 cubic centimeters. Find the dimensions of the box that minimizes the amount of material used.

(38) State the definition of the derivative and use it to compute the derivative of $f(x) := \sqrt{x}$.

(39) A particle moves along the number line with velocity $v(t) = t^2 - t + e^t$ meters per second.
(a) Find a function $s(t)$ which gives the position of this object if its initial position is 3 meters to the right of the origin.
(b) What is the average velocity of the object during the first 3 seconds?
(c) Find the acceleration $a(t)$.

(40)
   (a) State the Mean Value Theorem.
   (b) Use the Mean Value Theorem to explain why, for all a and b, that
$$|\sin(a) - \sin(b)| \leq |a - b|.$$
(41) Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.
(42) Sketch the graph of a function f which is continuous at $x = 1$ but is not differentiable at $x = 1$.
(43) The graph of a function f is as follows:



Find a $\delta > 0$ so that $|f(x) - 2| < .5$ whenever $0 < |x - 3| < \delta$.
(44) Prove using the ϵ, δ definition of limit that $\lim_{x\to 2}(x^2 - 1) = 3$.
(45) Find an overestimate and an underestimate for $\int_0^2 2^{x^2}\,dx$ using four subintervals of equal length.
(46) Sketch the graph of the function $f(x) = \dfrac{e^x}{x+1}$ (including everything).
(47) State both parts of the Fundamental Theorem of Calculus.
(48) Using the FTC, find $\left(\int_0^{2x} \sin(e^x)\,dx\right)'$.
(49) Using only a limit of Riemann sums, evaluate $\int_1^3 x^2\,dx$.
(50) Find the area between the curves $y = x^2 + 2x$ and $y = 3x$.
(51) Find the area between the curves $y^2 = 4 - x$ and $3y = x$.
(52) Where is the function $g(x) = \int_0^x \frac{(t+1)(t+2)}{t^4+1}\,dt$ increasing, and where is it decreasing? What is $g(0)$? Is $g(-1)$ positive or negative? It is true that $\lim_{x\to+\infty} g(x)$ exists (and is finite). You don't need to show that, but, is it positive, or negative? Why?