MATH 21, FALL, 2008, FINAL EXAM REVIEW SOLUTIONS

Suggestions. Imagine that you are working the problems on an exam with no help at all. Remember, a problem on the exam won't tell you what section of the book or notes it comes from, so it's important to spend some time working problems in random order. Work without a calculator since no calculators are allowed.

For the exam, you will be required to show all work. No calculators will be allowed. If you encounter an expression which is hard to simplify, leave it as it is. Please remember, however, to evaluate trigonometric, inverse trigonometric, logarithmic, and exponential functions at the given value whenever possible (for example $\sin(\pi/2)$).

Let Prof. Johnson know (david.johnson@lehigh.edu) if you think you may have found a mistake on any of these solutions. You should look at these solutions ***only*** after you have tried, hard, to solve the problems on your own, without looking at the book or other sources of help. The biggest difficulty during the exam is figuring out what method to apply – how to start working the problem. If you look at the solutions, even without looking through the details, then you will see the method used in the solution, which will make that problem easier to solve, but will not help you learn how to work the problems on your own.

(1) Evaluate the following limits, if they exist. If the limit does not exist, explain why.

(a) $\displaystyle\lim_{x\to -1} \frac{x^2-1}{x+1}$

Solution:

$$\begin{aligned}\lim_{x\to -1} \frac{x^2-1}{x+1} &= \lim_{x\to -1} \frac{(x-1)(x+1)}{x+1} \\ &= \lim_{x\to -1} \frac{(x-1)}{1} \\ &= -2.\end{aligned}$$

(b) $\displaystyle\lim_{x\to 1^-} \frac{x+2}{x-1}$

Solution:

$$\lim_{x\to 1^-} \frac{x+2}{x-1} = -\infty,$$

since the denominator is going to 0 and the numerator is bounded away from 0

(c) $\displaystyle\lim_{x\to \infty} \frac{x^2+1}{2x^2-1}$

Solution:

$$\begin{aligned}\lim_{x\to \infty} \frac{x^2+1}{2x^2-1} &= \lim_{x\to \infty} \frac{(x^2+1)\frac{1}{x^2}}{(2x^2-1)\frac{1}{x^2}} \\ &= \lim_{x\to \infty} \frac{1+\frac{1}{x^2}}{2-\frac{1}{x^2}} \\ &= \frac{1}{2}.\end{aligned}$$

(d) $\displaystyle\lim_{x\to 2^+} \frac{1}{\ln(x-2)}$

Solution:: This is not indeterminate, since the denominator is unabashedly headed to $-\infty$. So

$$\begin{aligned}\lim_{x\to 2^+}\frac{1}{\ln(x-2)} &= \frac{1}{-\infty}\\ &= 0.\end{aligned}$$

(e) $\lim_{x\to 0^+}\sqrt{x}\ln x$

Solution:: This one uses l'Hôpital's Rule, ∞/∞ form

$$\begin{aligned}\lim_{x\to 0^+}\sqrt{x}\ln x &= \lim_{x\to 0^+}\frac{\ln x}{\left(\frac{1}{\sqrt{x}}\right)}\\ &= \lim_{x\to 0^+}\frac{\ln x}{\left(\frac{1}{\sqrt{x}}\right)}\\ &= \lim_{x\to 0^+}\frac{\frac{1}{x}}{\left(\frac{-1}{2x^{3/2}}\right)}\\ &= \lim_{x\to 0^+}\frac{-2x^{3/2}}{x}\\ &= \lim_{x\to 0^+} -2x^{1/2}\\ &= 0.\end{aligned}$$

(f) $\lim_{x\to 0}\frac{\sin x}{3x}$

Solution: This doesn't really use l'Hôpital's Rule, although you could use it. It really uses the fact that $\lim_{x\to 0}\frac{\sin x}{x}=1$, and so

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{3x} &= \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x}\\ &= \frac{1}{3}.\end{aligned}$$

(g) $\lim_{x\to\infty}\frac{\sqrt{2x^2+5x+1}}{x}$

Solution:: This can use l'Hôpital's Rule, as well, but it makes more sense without it (you basically have the same work to do with l'Hôpital's Rule as without it)

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{2x^2+5x+1}}{x} &= \lim_{x\to\infty}\frac{\left(\sqrt{2x^2+5x+1}\right)\frac{1}{x}}{(x)\frac{1}{x}}\\ &= \lim_{x\to\infty}\frac{\left(\sqrt{2+\frac{5}{x}+\frac{1}{x^2}}\right)}{1}\\ &= \sqrt{2}.\end{aligned}$$

(h) $\lim_{x\to 2}\frac{|x-2|}{x-2}$

Solution:: This limit does not exist. From the right, the fraction is 1, but from the left, the fraction is -1, so there can be no limit at 2.

(i) $\lim_{x\to 1^-}\arcsin x$

Solution::

$$\lim_{x\to 1^-}\arcsin x = \pi/2,$$

which follows from the definition.

(j) $\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x$

Solution:: This again can be solved with the help of our French friend, but can be done without it as well. Since

$$\begin{aligned}
\ln\left(\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x\right) &= \lim_{x\to\infty}\ln\left(\left(1+\frac{r}{x}\right)^x\right)\\
&= \lim_{x\to\infty} x\left(\ln\left(1+\frac{r}{x}\right)\right)\\
&= \lim_{x\to\infty}\frac{\ln\left(1+\frac{r}{x}\right)}{\frac{1}{x}}\\
&= r\lim_{x\to\infty}\frac{\ln\left(1+\frac{r}{x}\right)}{\frac{r}{x}}.
\end{aligned}$$

Now, set $h=\frac{r}{x}$; and we continue the above, with the observation that $\ln(1)=0$:

$$\begin{aligned}
&= r\lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}\\
&= r\,(\ln(x))'\big|_{x=1}\\
&= r\cdot 1.
\end{aligned}$$

So, remembering that we had taken the logarithm to get here,

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = e^r.$$

(k) $\lim_{h\to 0}\frac{4^h-1}{h}$

Solution: This is best explained in terms of the derivative of exponentials,

$$\lim_{h\to 0}\frac{4^h-1}{h} = f'(0),$$

where $f(x)=4^x$. But, since $4^x=e^{x\ln(4)}$ by the rules of logarithms,

$$\begin{aligned}
\lim_{h\to 0}\frac{4^h-1}{h} &= f'(0)\\
&= \frac{d}{dx}\bigg|_{x=0} e^{x\ln(4)}\\
&= \ln(4)\,(e^x)'\big|_{x=0}\\
&= \ln(4)\cdot 1.
\end{aligned}$$

(l) $\lim_{x\to 0}\frac{\arctan x}{e^x-1}$

Solution: This uses l'Hôpital's Rule, $\frac{0}{0}$ form;

$$\begin{aligned}
\lim_{x\to 0}\frac{\arctan x}{e^x-1} &= \lim_{x\to 0}\frac{\left(\frac{1}{1+x^2}\right)}{e^x}\\
&= 1.
\end{aligned}$$

(m) $\lim_{x\to\infty} x-\frac{x^2+1}{x+1}$

Solution:: This should be simplified first,

$$\begin{aligned}\lim_{x\to\infty} x - \frac{x^2+1}{x+1} &= \lim_{x\to\infty} \frac{x\,(x+1) - \left(x^2+1\right)}{x+1} \\ &= \lim_{x\to\infty} \frac{x-1}{x+1} \\ &= 1.\end{aligned}$$

(n) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{6i}{n}\right)+1\right]$ This is an integral, given as a limit of sums. What integral?

Solution: This is the integral $\int_0^2 (3x+1)dx = 8$.

(2) Evaluate the following

(a) $\displaystyle\frac{d}{dx}(e^{-2x} + 3\,\ln|x|)\Big|_{x=-1/2}$

Solution:

$$\begin{aligned}\frac{d}{dx}(e^{-2x} + 3\,\ln|x|)\Big|_{x=-1/2} &= \left(-2e^{-2x} + \frac{3}{x}\right)\Big|_{x=-1/2} \\ &= -2e - 6.\end{aligned}$$

(b) $(\csc x/(1+x^2))'$

Solution: Since $(\csc(x))' = -\csc(x)\cot(x)$, by the quotient rule we have

$$(\csc x/(1+x^2))' = \frac{-\csc(x)\cot(x)(1+x^2) - 2x\csc(x)}{(1+x^2)^2}.$$

(c) $\displaystyle\frac{d}{dx}\ln|3x-2|$

Solution:

$$\frac{d}{dx}\ln|3x-2| = \frac{3}{3x-2}$$

(d) $\displaystyle\frac{d}{dx}\sin^2(\cos(1+x^2))$

Solution: Use the chain rule, thinking of $\sin^2(\cos(1+x^2)) = \left(\sin(\cos(1+x^2))\right)^2$ — so, 4 layers.

$$\frac{d}{dx}\sin^2(\cos(1+x^2)) = -4x\sin(\cos(1+x^2))\cos(\cos(1+x^2))\sin(1+x^2).$$

(e) $\displaystyle\frac{d}{dx}(\arctan(x))^2\Big|_{x=-1}$

Solution:

$$\begin{aligned}\frac{d}{dx}(\arctan(x))^2\Big|_{x=-1} &= 2(\arctan(x))\frac{1}{1+x^2}\Big|_{x=-1} \\ &= -\frac{\pi}{4}.\end{aligned}$$

(f) $\displaystyle\frac{d}{dx}\left(\int_1^{x^2}\sin^3(t)\,dt\right)$

Solution: This is one of those trick problems with the FTC: if $F(x) = \int_1^x \sin^3(t)\,dt$, then for one, $F'(x) = \sin^3(x)$, and

$$\begin{aligned}\frac{d}{dx}\left(\int_1^{x^2}\sin^3(t)\,dt\right) &= \frac{d}{dx}F(x^2) \\ &= \sin^3(x^2)2x.\end{aligned}$$

(g) $(x^{x^2+x})'$

Solution: The big problem here is unraveling what this is:

$$(x^{x^2+x}) = e^{(\ln(x))(x^2+x)},$$

so

$$\begin{aligned}(x^{x^2+x})' &= \left(e^{(\ln(x))(x^2+x)}\right)' \\ &= e^{(\ln(x))(x^2+x)}\left(\frac{1}{x}(x^2+x)+\ln(x)(2x+1)\right).\end{aligned}$$

(3) Suppose that $f(x)$ and $g(x)$ are differentiable functions so that $f(1)=2$, $f'(1)=3$, $f(4)=2$, $f'(4)=3$, $f(5)=-3$, $f'(5)=1$, $g(1)=4$, $g'(1)=5$, $g(2)=1$, and $g'(2)=3$. Find $\frac{d}{dx}\left[f(g(x))\right]|_{x=1}$.

Solution: By the chain rule,

$$\begin{aligned}\frac{d}{dx}\left[f(g(x))\right]|_{x=1} &= f'(g(1))g'(1) \\ &= f'(4)\cdot 5 \\ &= 3\cdot 5 = 15.\end{aligned}$$

(4) Suppose that $f(x)$ and $g(x)$ are differentiable functions so that $f(1)=1$, $f'(1)=4$, $f(5)=3$, $f'(5)=2$, $f(4)=-3$, $f'(4)=1$, $g(1)=5$, $g'(1)=4$, $g(2)=1$, and $g'(2)=3$. Find $\frac{d}{dx}\left[f(g(x))\right]|_{x=1}$

Solution: By the chain rule, (Be careful to use the new list of values)

$$\begin{aligned}\frac{d}{dx}\left[f(g(x))\right]|_{x=1} &= f'(g(1))g'(1) \\ &= f'(5)\cdot 4 \\ &= 2\cdot 4 = 8.\end{aligned}$$

(5) Assume that $y=f(x)$ satisfies the equation

$$y^5+3x^2y^2+5x^4=12.$$

Find dy/dx in terms of x and y.

Solution: *Here you use implicit differentiation.*

$$\left(y^5+3x^2y^2+5x^4\right)'=(12)'=0,$$

so, using the chain rule and remembering that $y=y(x)$ is a function of x,

$$\begin{aligned}0 &= 5y^4\frac{dy}{dx}+3(2xy^2+x^2 2y\frac{dy}{dx})+20x^3 \\ &= \left(5y^4+6x^2y\right)\frac{dy}{dx}+6xy^2+20x^3.\end{aligned}$$

Solving for dy/dx gives

$$\frac{dy}{dx}=\frac{-\left(6xy^2+20x^3\right)}{5y^4+6x^2y}.$$

(6) If $x^2-3xy+y^2=5$, find $\frac{dy}{dx}$ at the point (1,-1). *(10 points).*

Solution: Again, use implicit differentiation, thinking of y as a function of x:

$$\begin{aligned}(5)' &= \left(x^2-3xy+y^2\right)' \\ 0 &= 2x-3y-3x\frac{dy}{dx}+2y\frac{dy}{dx} \\ &= 2x-3y-(3x-2y)\frac{dy}{dx},\end{aligned}$$

so

$$\begin{aligned}(3x-2y)\frac{dy}{dx} &= 2x-3y \\ \frac{dy}{dx} &= \frac{2x-3y}{3x-2y}.\end{aligned}$$

At the point (1,-1), $\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{2\cdot 1-3(-1)}{3\cdot 1-2(-1)} = 1$.

(7) Find an equation of the tangent line to the graph of $x^3-xy-y^2+5=0$ at $(1,2)$.

Solution: Implicit differentiation, once more. Differentiate both sides:

$$\begin{aligned}0' &= \left(x^3-xy-y^2+5\right)' \\ 0 &= 3x^2-y-x\frac{dy}{dx}-2y\frac{dy}{dx}+0 \\ &= 3x^2-y-\frac{dy}{dx}(x+2y),\end{aligned}$$

so $\dfrac{dy}{dx} = \dfrac{3x^2-y}{x+2y}$. At the point $(1,2)$, $\dfrac{dy}{dx} = \dfrac{3-2}{1+4} = \dfrac{1}{5}$. Now, that finds the slope of the curve at that point. To find the tangent line, plug into the standard point-slope form of the equation of a line:

$$\frac{y-2}{x-1} = \frac{1}{5},$$

or $y = \dfrac{1}{5}x+\dfrac{9}{5}$.

(8) Show, using the definition of the inverse tangent and implicit differentiation, that

$$\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}.$$

Solution: If $\theta = \tan^{-1}(x)$, then $\tan(\theta) = x$. So, think of a right triangle with opposite side (to the angle θ) x and base 1. It has hypotenuse $\sqrt{1+x^2}$, so $\cos(\theta) = 1/\sqrt{1+x^2}$. But, then, since $\tan(\theta) = x$,

$$\begin{aligned}\frac{d}{dx}\left(\tan(\theta)\right) &= 1, \text{ or} \\ \sec^2(\theta)\frac{d\theta}{dx} &= 1.\end{aligned}$$

But then,

$$\begin{aligned}1 &= \sec^2(\theta)\frac{d\theta}{dx}, \text{ or} \\ \cos^2(\theta) &= \frac{d\theta}{dx} \\ \frac{1}{1+x^2} &= \frac{d\theta}{dx} \\ \frac{1}{1+x^2} &= \frac{d\tan^{-1}(x)}{dx}.\end{aligned}$$

(9) The population of certain bacteria grows at a rate proportional to its size. It increases by 60% after 3 days. How long does it take for the population to double? (12 points)

Solution: Since the population of the bacteria grows at a rate proportional to its size, if $P(t)$ is the population, then $P'(t) = kP(t)$, which means that $P(t) = Ae^{kt}$ for some numbers A and k. This is the solution of that general differential equation. To fit this solution to the particular data, we have that $P(0) = A$ (the initial population, which does not need to be a specific number), and $P(3) = 1.6A$, 60% more than we initially had, after 3 days. But then $P(t) = Ae^{kt}$ with the same A as the initial population (at time 0), and

$$\begin{aligned} 1.6A &= P(3) \\ &= Ae^{k3}. \end{aligned}$$

So. $1.6 = e^{3k}$, or, taking natural logs of both sides, $\ln(1.6) = 3k$, or $k = \frac{1}{3}\ln(1.6)$. Then, $P(t)$ is given more explicitly by

$$P(t) = Ae^{\frac{1}{3}\ln(1.6)t}.$$

The time T it takes for the population to double satisfies

$$\begin{aligned} 2A &= P(T) \\ &= Ae^{\frac{1}{3}\ln(1.6)T}, \end{aligned}$$

so, taking ln() of both sides again, $\ln(2) = \ln(e^{\frac{1}{3}\ln(1.6)T}) = \frac{1}{3}\ln(1.6)T$, so $T = \frac{3\ln 2}{\ln(1.6)}$.

(10) A new radioactive substance, Doublemintium (D_m), has been found sticking to the undersides of the seats in Packard Auditorium. 70 grams of pure D_m was collected initially. 5 days later only 60 grams of the stuff was still D_m; the rest had decayed into lead and tar. Assuming that the rate of decay of D_m is proportional to the amount present, how much will there be after 20 days?

Solution: Since the amount of D_m present in the sample decays away at a rate proportional to the amount present, if $y(t)$ is the amount of D_m present at time t, then $y'(t) = ky(t)$, which means that $y(t) = Ae^{kt}$ as with a population problem. since 70g was initially found, $A = 70$, and so $y = 70e^{kt}$. But also,

$$\begin{aligned} 60 &= y(5) \\ &= 70e^{k5}, \end{aligned}$$

and so $\frac{6}{7} = e^{5k}$, or $k = \frac{\ln(6/7)}{5}$. Then, the amount present after 20 days will be $y(20)$,

$$\begin{aligned} y(20) &= 70e^{k20} \\ &= 70e^{4\ln(6/7)} \\ &= 70\left(\frac{6}{7}\right)^4. \end{aligned}$$

(11) I bought a cup of coffee at McBurger's. It was far too hot to drink, 90^o Celsius. After 10 minutes, the coffee is at 80^o. The air in McBurger's is kept at an air-conditioned constant of 25^o. How long will I have to wait until the coffee is 70^o and thus cool enough to drink?

Solution: It's best to solve this in terms of the function $y = T - A$, the temperature of the coffee minus the ambient temperature. In that form, since Newton's law of cooling says that the rate of change of temperature of an object is proportional to the difference in temperature between the object and the ambient temperature, then $T' = (T - A)' = y'$ satisfies $y' = ky$, so $y = Be^{kt}$ for some numbers B and k. In this problem $A = 25$. The rest of the problem simply uses the two temperatures at the two times to find the constants B and k. Since $y(0) = 90 - 25 = 65$ is the difference in temperatures between the coffee and the room at time 0, $65 = y(0) = Be^0 = B$. Then, since $55 = y(10)$, $55 = Be^{k10} = 65e^{k10}$, so $\ln(55/65) = k10$, or $k = \frac{\ln(11/13)}{10}$. Then, to answer the final question, set $T = 70$ and solve for t:

$$\begin{aligned} 70 - 25 &= y(t) \\ 45 &= 65e^{(kt)} \\ &= 65e^{\left(\frac{t\ln(11/13)}{10}\right)}. \end{aligned}$$

So $\ln(9/13) = \frac{t\ln(11/13)}{10}$, or $t = \frac{10\ln(9/13)}{\ln(11/13)}$. As tempting as it may be, you can't cancel those 13's.

(12) A spherical snowball melts in such a way that its volume decreases at the rate of 2 cubic centimeters per minute. At what rate is the radius decreasing when the volume is 400 cubic centimeters?

Solution: The volume of the snowball is $V = \frac{4}{3}\pi r^3$, which is an equation relating the volume to the radius. Differentiate both sides, as functions of t, to see how the rates of change are related: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. You also are given that $\frac{dV}{dt} = 2$. When $V = 400$, then $400 = \frac{4}{3}\pi r^3$, or $r = \left(\frac{300}{\pi}\right)^{1/3}$. At that instant, then $\frac{dV}{dt} = 4\pi\left(\frac{300}{\pi}\right)^{2/3}\frac{dr}{dt}$, so $2 = 4\pi\left(\frac{300}{\pi}\right)^{2/3}\frac{dr}{dt}$, and finally $\frac{dr}{dt} = \frac{2}{4\pi\left(\frac{300}{\pi}\right)^{2/3}} = \frac{1}{2(300)^{2/3}\pi^{1/2}}$.

(13) Egbert is flying his kite. It is 75 feet off the ground, moving horizontally away from Egbert at a rate of 5 feet per second. How fast is Egbert letting out the string when the kite is 100 feet (horizontally) downwind of him?

Solution:: If x is the horizontal distance from Egbert to his kite, and s is the amount of string he has played out, then we know that $dx/dt = +5$, and we want to know ds/dt when $x = 100$. Because of the Pythagorean theorem, $x^2 + 75^2 = s^2$ is the equation relating x to s, which we then differentiate to get a relationship between the rates of change.

$$\begin{aligned} x^2 + 75^2 &= s^2, \text{ so} \\ 2\,x\,\frac{dx}{dt} + 0 &= 2\,s\,\frac{ds}{dt}. \end{aligned}$$

When $x = 100$ $(4\cdot 25)$, then $s = 125 = 5\cdot 25$ since the other leg is $75 = 3\cdot 25$; this is a $3-4-5$ triangle. Plugging in,

$$2\,100\,5 = 2\,125\,\frac{ds}{dt},$$

or

$$\frac{ds}{dt} = \frac{500}{125} = 4.$$

which is the rate at which the line is being played out.

(14) A 20 foot ladder is leaning against a wall. A painter stands on the top of the ladder, minding his own business. Some fool comes by and ties his dog to the base of the ladder, a cat comes along, and the dog chases after the cat, dragging the base of the ladder with him at a rate of 2 feet per second directly away from the wall. How fast is the painter falling when he is 12 feet from the ground?

Solution: This is a fairly standard related-rates problem. Set x to be the distance from the base of the ladder to the wall, and set y to be the height of the painter above the ground. We **know** that $\frac{dx}{dt} = +2$ and we **want** to know $\frac{dy}{dt}$ **when** $y = 12$. Then, since the ladder has constant length 20 until it smashes against the ground, the Pythagorean theorem tells us that $x^2 + y^2 = 20^2$, which is an equation between what we know and what we want to know. Differentiate both sides of that equation with respect to time to get the relationship between their rates of change, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. At "the when", you have $y = 12$, so plug into the equation $x^2 + y^2 = 20^2$ to see that $x^2 + 12^2 = 20^2$ at that instant, so $x = 16$ (it's a 3-4-5 triangle). Then, you plug these values into the equation relating the rates, and solve for $\frac{dy}{dt}$:

$$\begin{aligned} 0 &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2\cdot 16\cdot 2 + 2\cdot 12\cdot\frac{dy}{dt}, \end{aligned}$$

or $\frac{dy}{dt} = -\frac{32}{12} = -\frac{8}{3}$, meaning that the painter is falling at 8/3 feet per second at that instant.

(15) Egbert is drinking a daiquiri (non-alcoholic) out of a glass that is a cone with a radius at the top of 3 inches and a height of 5 inches. He drinks his daiquiri at a constant rate of 2 cubic inches per second (through a straw). How fast is the top of the daiquiri falling when it is 4 inches above the bottom of the glass?

Solution: We **know** that the rate of change of the volume V is -2 (in cubic inches per second), $\frac{dV}{dt} = -2$. We **want** to know $\frac{dh}{dt}$, where h is the height of the liquid in his glass, **when** $h = 4$.

For this one you need to know what the volume of a cone is. If the base-radius (the base is at the top for this, since in order to hold liquid the glass has to open upwards) is r and the height (from base to apex, which here is pointing downward) is h, the volume is $V = \frac{1}{3}\pi r^2 h$. Now, before you think about taking $r = 3$ or $h = 5$, remember that the glass is only partially full. Even at the time when the daiquiri is 4 inches deep, there is an inch-high gap between the top of the liquid and the top of the glass. So, why mention those numbers? The cone of the entire glass is similar to the cone of the daiquiri in the glass. So, the right triangle formed by the center line from the apex to the top of the liquid to the edge of the glass, then back down to the apex along the glass is a similar triangle to the one formed by the center line from the apex to the top of the glass, then to the edge and back down. The larger triangle has height 5 and base 3, and the smaller has height h and base r. Since the ratios of corresponding sides are the same for similar triangles (or similar cones), $\frac{5}{3} = \frac{h}{r}$, or $r = \frac{3}{5}h$. With that observation, now the volume satisfies $V = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h = \frac{3\pi}{25}h^3$. This gives an equation relating the variable V we know something about to the variable h we want to know something about. We then differentiate both sides with respect to time, $\frac{dV}{dt} = \frac{9\pi}{25}h^2\frac{dh}{dt}$, plug in for the fact that $\frac{dV}{dt} = -2$ and (at "the when") $h = 4$ [Careful, don't plug in until after you have differentiated], so

$$\begin{aligned} \frac{dV}{dt} &= \frac{9\pi}{25}h^2\frac{dh}{dt} \\ -2 &= \frac{9\pi}{25}4^2\frac{dh}{dt}, \end{aligned}$$

or $\frac{dh}{dt} = -\frac{25}{72\pi}$, the height of liquid in the glass is falling at a rate of $25/(72\pi)$ inches per second.

(16) A man, walking at night, is walking directly towards a streetlight. The light is 10 feet off the ground, and the man is 6 feet tall and walking at 4 feet per second. When the man is 8 feet from the streetlight, how fast is the length of his shadow changing?

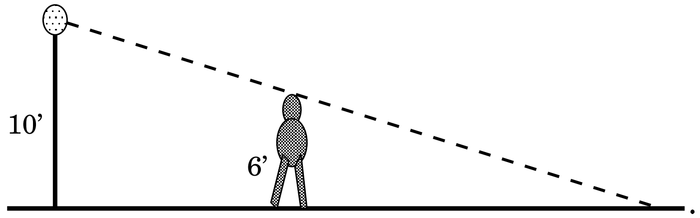
Solution:: Here similar triangles gives the relationship between the variables. Since the big triangle with vertical leg the lightpost is similar to the smaller one (with the man as the vertical leg), we have, if x is the distance from the man to the pole, and y is the length of the shadow, then

$$\frac{6}{y} = \frac{10}{x+y},$$

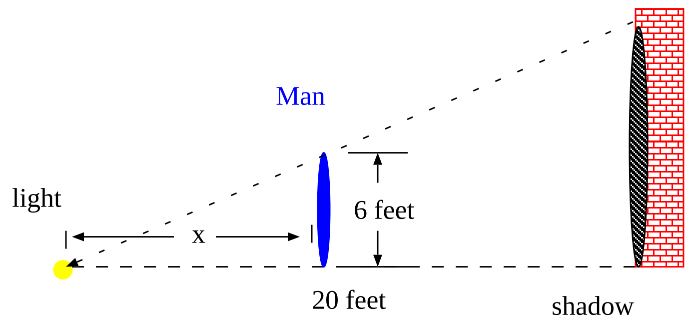
or $6(x+y) = 10y$, or finally, $6x = 4y$. Thus, since you know $\frac{dx}{dt} = -4$, we get immediately that

$$\begin{aligned} 6\frac{dx}{dt} &= 4\frac{dy}{dt}, \text{ or} \\ 6 \cdot (-4) &= 4\frac{dy}{dt}, \end{aligned}$$

so $\frac{dy}{dt} = -6$, which doesn't depend upon x so you don't have to plug in $x = 8$ anywhere. The shadow is shrinking at 6ft/sec. Here is the picture



(17) The side of a tall building is illuminated by a floodlight mounted on the lawn in front of the building. The floodlight is at ground level, and is 20 feet from the building. A man, 6 feet tall, is walking towards the floodlight at 4 feet/sec. How fast is his shadow on the wall growing when he is 10 feet from the floodlight?



Solution: Looking at the picture, the distance labeled x is the distance from the man to the light. Let's also label as y the height of the shadow on the wall. You **know** that $\frac{dx}{dt} = -4$, and you **want** $\frac{dy}{dt}$ **when** $x = 10$. Similar triangles works as for the earlier problem, but a bit differently. The large triangle (light to shadow) has base 20 and height y, but the small triangle has base x and height 6, so similar triangles gives $\frac{y}{20} = \frac{6}{x}$, which is, as we need, an equation relating what we know to what we want. Differentiating both sides of that equation with respect to time gives

$$\frac{1}{20}\frac{dy}{dt} = -\frac{6}{x^2}\frac{dx}{dt},$$

and plugging in $\frac{dx}{dt} = -4$ and (at the time in question) $x = 10$,

$$\begin{aligned}\frac{1}{20}\frac{dy}{dt} &= -\frac{6}{x^2}\frac{dx}{dt}\\ \frac{1}{20}\frac{dy}{dt} &= -\frac{6}{100}(-4),\end{aligned}$$

or $\frac{dy}{dt} = +4.8$. So, the shadow is growing at 4.8 ft/sec.

(18) Use differentials to approximate

$$\sqrt{24}.$$

Solution:

$$\sqrt{24} = f(24)$$

where

$$f(x) = \sqrt{x}.$$

Now, linear approximation says that $f(x) \approx f(a) + f'(a)(x - a)$. Take $a = 25$ since it is easy to evaluate square roots there, and then $\Delta x = -1$, or

$$f(x) = \sqrt{x} \approx 5 + \frac{1}{2 \cdot 5}(x - 25).$$

Taking $x = 24$ gives

$$f(24) = \sqrt{24} \approx 5 - \frac{1}{10} = 4.9.$$

(19) Given that the function f satisfies $1 \leq f(x) \leq x^2 + 2x + 2$ for all x, find $\lim_{x \to -1} f(x)$.

Solution: Since $f(x)$ is trapped between 1 and $x^2 + 2x + 2$, so must its limit be at -1 . But, at $x = -1$, $\left(x^2 + 2x + 2\right)\big|_{-1} = 1$, and $x^2 + 2x + 2$ is of course continuous, so it's limit at -1 will be 1, so by the squeeze principal $\lim_{x \to -1} f(x)$ is trapped between 1 and 1. Thus $\lim_{x \to -1} f(x) = 1$.

(20) Does the equation $x^3 - x^2 = 3$ have a solution in the interval $1 < x < 2$? Justify your answer.

Solution: Well, If $f(x) = x^3 - x^2$, then $f(1) = 0$ and $f(2) = 4$, so by the IVT any value in between 0 and 4, such as 3, is hit by $f(x)$, so, yes, $f(x) = 3$, that is $x^3 - x^2 = 3$, does have at least one solution in $[1, 2]$.

(21) Let $f(x) = x^2 + 2x$. Find $f'(1)$ using only the limit definition of the derivative.

Solution:

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{((1+h)^2 + 2(1+h)) - 3}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2 + 2h}{h} \\ &= 4. \end{aligned}$$

(22) Below is the graph of a function f defined on the open interval $0 < x < 4$. Find those values of c with $0 < c < 4$ for which

(a) $\lim_{x \to c} f(x)$ does *not* exist.

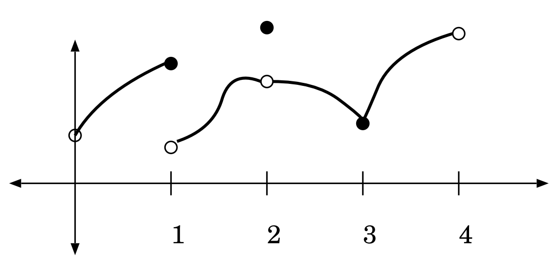
Solution: $c = 1$ only.

(b) f is *not* continuous at c.

Solution: $c = 1$ and $c = 2$.

(c) f is *not* differentiable at c.

Solution: $c = 1$, $c = 2$, and $c = 3$.



(23) Find an equation for the tangent line to the graph of $x^3 - xy - y^2 + 5 = 0$ at $(1, 2)$.

Solution: The slope, by implicit differentiation, is

$$3x^2 - y - x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = \frac{3x^2 - y}{x + 2y}.$$

At $(1,2)$, we get
$$\frac{dy}{dx} = \frac{3-2}{1+4} = \frac{1}{5},$$
and so the tangent line is the line which goes through $(1,2)$ with slope $\frac{1}{5}$, so has equation
$$\frac{y-2}{x-1} = \frac{1}{5},$$
or $y = \frac{1}{5}x + \frac{9}{5}$.

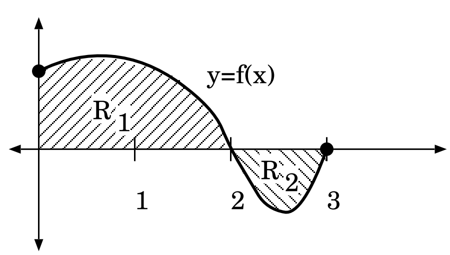
(24) Estimate $\sqrt{15.5}$ using differentials.

Solution: Let $f(x) = \sqrt{x}$, and take $a = 16$. Then, by linear approximation (AKA differentials),
$$\begin{aligned} f(15.5) &= f(16 + (-\frac{1}{2})) \\ &= f(16) + f'(16)(-\frac{1}{2}) \\ &= 4 + \frac{1}{2 \cdot 4}(-\frac{1}{2}) \\ &= \frac{63}{16}. \end{aligned}$$

(25) A spherical balloon is being inflated with air at the rate of 2 cubic inches per second. At what rate is the radius expanding when when the volume is 20π cubic inches?

Solution: Since the volume and the radius are related by the equation $V = \frac{4}{3}\pi r^3$, their derivatives are related by $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Since $\frac{dV}{dt} = 2$, and since, when $V = 20\pi$, $20\pi = V = \frac{4}{3}\pi r^3$, or $15 = r^3$, $r = (15)^{1/3}$, and so
$$2 = \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(15^{2/3})\frac{dr}{dt},$$
or
$$\frac{dr}{dt} = \frac{2}{4\pi(15^{2/3})} = \frac{1}{2\pi(15^{2/3})}.$$

(26) The graph of a continuous function f determines two regions R_1 and R_2 as indicated below. The region R_1 has area $3/2$ and the region R_2 has area $5/8$. Find $\int_0^3 f(x)\,dx$.



Solution:: The integral will be $\frac{3}{2} - \frac{5}{8} = \frac{7}{8}$.

(27) Evaluate

(a) $\displaystyle\int \left(2 + \frac{1}{\sqrt{x}} + 2e^{-3x}\right) dx$

Solution: $= 2x + 2\sqrt{x} - \frac{2}{3}e^{-3x} + c$.

(b) $\displaystyle\int \left(\frac{2}{1+x^2} - 3\sin x\right) dx$

Solution: $= 2\tan^{-1}(x) + 3\cos(x) + c$.

(c) $\int x\sqrt{1+x^2}\,dx$

Solution: Set $u = 1 + x^2$. Then, $du = 2xdx$, and so

$$\begin{aligned}\int x\sqrt{1+x^2}\,dx &= \int \sqrt{u}\left(\frac{1}{2}du\right)\\ &= \left(\frac{u^{3/2}}{3/2}\right)\frac{1}{2}+c\\ &= \frac{1}{3}(1+x^2)^{3/2}+c.\end{aligned}$$

(d) $\int_1^3 \left(x^2 - \frac{1}{x}\right)\,dx$

Solution: $= \frac{1}{3}x^3 - \ln|x|\big|_1^3 = (9 - \ln(3)) - \frac{1}{3} = \frac{26}{3} - \ln(3)$.

(28) Below is the graph of a function f defined on the open interval $0 < x < 4$. Find

(a) The critical numbers

Solution: Since the endpoints are not part of the domain, they would not count as critical numbers (AKA critical points). So, the only such points are when $x = 1$, where the derivative is 0, and $x = 2$, where the derivative does not exist.

(b) The local extrema

Solution: Again, the endpoints cannot count since they are excluded from the domain. So, the only local extrema are at $x = 1$ and $x = 2$.

(c) The absolute extrema

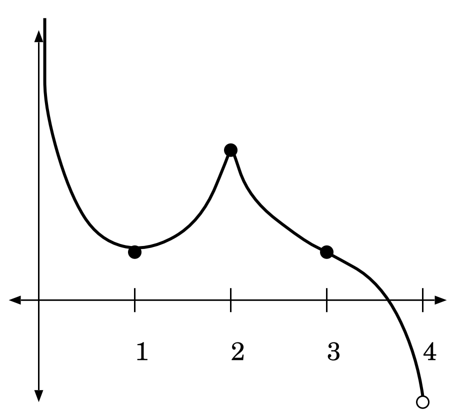
Solution: There are none.

(d) Where the graph is concave up and where concave down.

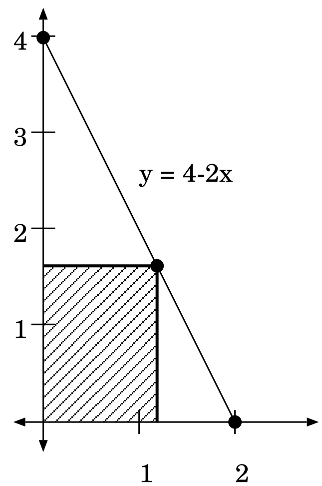
Solution: Concave up on $(0, 2]$, and $[2, 3]$ (technically, you should include the endpoints, but it doesn't really matter much.

(e) The inflection points.

Solution: Only at $x = 3$.



(29) Find the largest possible area of a rectangle inscribed in the triangle bounded by $x = 0, y = 0$, and $y = 4 - 2x$ as indicated below. Please verify that your answer is really the maximum.



Solution: The largest rectangle inside the triangle is indeed as pictured, with one vertex on the hypotenuse, and another at the origin. The area of such a rectangle is $A = xy$, where (x, y) are the coordinates of the point of the rectangle that touches the hypotenuse of the triangle. That is what we are supposed to maximize. But also, the constraint is reasonably that (x, y) really does line on the hypotenuse, rather than insides, so $y = 4 - 2x$, giving the constraint between the two points, and the area, as function of x, is $A(x) = A = xy = x(4 - 2x) = 4x - 2x^2$, and $\frac{d}{dx}A(x) = 4 - 4x$ with only one critical point, at $x = 1$. The domain of $A(x)$ is $0 \leq x \leq 2$, and $A(0) = A(2) = 0$, so the maximum has to be at $x = 1$.

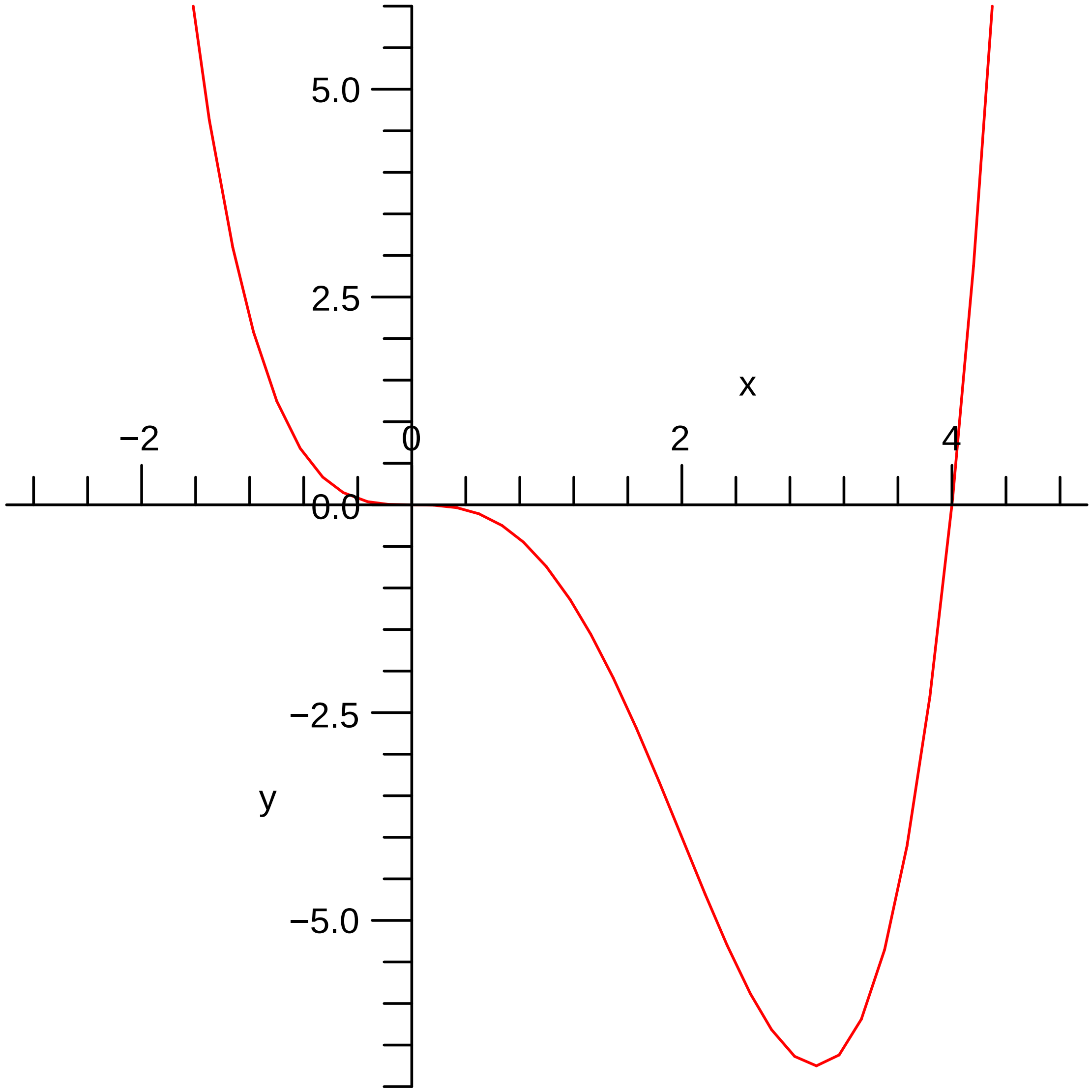
(30) Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.

Solution: The curves intersect when $x^2 = x + 2$, or $x^2 - x - 2 = 0$, which occurs at $x = -1$, and $x = 2$. So, the area (since the top curve is the line) is

$$\begin{aligned} A &= \int_{-1}^{2} \left(x + 2 - x^2\right) dx \\ &= \frac{x^2}{2} + 2x - \frac{x^3}{3}\bigg|_{-1}^{2} \\ &= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) \\ &= 6 - 3 - \frac{1}{2} + 2 \\ &= \frac{9}{2}. \end{aligned}$$

(31) Sketch the graph of the function $f(x) = \frac{1}{4}x^4 - x^3$. Include (with labels) all local extreme points, all absolute extreme points, all inflection points, and all intercepts.

Solution:: We start with the domain, which is clearly the entire line. We then find the intercepts, which are only at $(0, 0)$ and $(4, 0)$. There are no asymptotes or symmetry. We then look at the derivatives: $f'(x) = x^3 - 3x^2 = x^2(x - 3)$, so f has critical points at $x = 0, 3$, with values $f(0) = 0$ and $f(3) = -27/4$. $f'(x) > 0$ for $x > 3$, and $f' < 0$ for $x < 0$ or $0 < x < 3$, so it levels off only at $x = 0$, but has a local minimum at $x = 3$. For the second derivative, $f'' = 3x^2 - 6x = 3x(x - 2)$, which gives inflection points when $x = 0, 2$, so the points are $(0, 0)$ and $(2, -4)$, and for $x > 2$ or $x < 0$, $f''(x) > 0$, but if $0 < x < 2$, $f''(x) < 0$. Here is the drawing (which I cheated on and generated with Maple):



(32) Find $f'(x)$ for

(a) $f(x) = 4x^3 - x + 1$

Solution: $f'(x) = 12x^2 - 1$

(b) $f(x) = \dfrac{1}{x^2 + \sqrt{x}}$

Solution: $f'(x) = \dfrac{-(2x + \frac{1}{2\sqrt{x}})}{(x^2 + \sqrt{x})^2}$.

(c) $f(x) = \dfrac{x+1}{x-1}$

Solution: $f'(x) = \dfrac{1(x-1) - 1(x+1)}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$.

(d) $f(x) = \dfrac{2}{x^5}$

Solution: $f'(x) = 2(-5)x^{-6} = -\frac{10}{x^6}$.

(e) $f(x) = \ln|x| - 3e^{2x}$

Solution: $f'(x) = \frac{1}{x} - 6e^{2x}$.

(f) $f(x) = \tan^{-1} x \sin^{-1} x$

Solution: Using first the product rule, then the formulas for the derivatives of the inverse sine and tangent, $f'(x) = \frac{1}{1+x^2}\sin^{-1}(x) + \tan^{-1}(x)\frac{1}{\sqrt{1-x^2}}$

(g) $f(x) = \cos(x^2)$
Solution: $f'(x) = -2x\sin(x^2)$.

(h) $f(x) = x^x$
Solution: Since $f(x) = e^{x\ln(x)}$, $f'(x) = e^{x\ln(x)}(\ln(x)+1) = x^x(\ln(x)+1)$.

(i) $f(x) = \int_5^x \sqrt{1+t^3}\,dt$
Solution:: $f'(x) = \sqrt{1+x^3}$, by a direct application of the FTC, part I.

(33) Evaluate the following.

(a) $\int (x^3 + x^{1/3})\,dx$
Solution: $= \frac{1}{4}x^4 + \frac{3}{4}x^{4/3} + c$.

(b) $\int \sinh(x)\cosh(x)\,dx$
Solution: Use substitution with $u = \sinh(x)$, and $\int \sinh(x)\cosh(x)\,dx = \int udu = \frac{1}{2}\sinh^2(x) + c$.

(c) $\int x^2 e^{x^3}\,dx$
Solution: Use u-substitution. $u = x^3$, then $du = 3x^2dx$, and

$$\begin{aligned}\int x^2 e^{x^3}\,dx &= \int e^u \frac{1}{3}du \\ &= \frac{1}{3}e^{x^3} + c.\end{aligned}$$

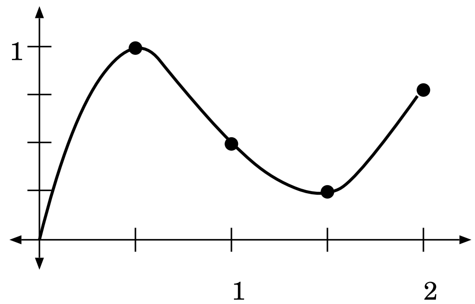
(d) $\int \frac{1+x}{1+x^2}\,dx$
Solution: This has to be broken up into two integrals. The first is standard, but the second needs the u-substitution $u = 1+x^2$, with $du = 2xdx$:

$$\begin{aligned}\int \frac{1+x}{1+x^2}\,dx &= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx \\ &= \tan^{-1}(x) + \int \frac{\frac{1}{2}du}{u} \\ &= \tan^{-1}(x) + \frac{1}{2}\ln|1+x^2| + c.\end{aligned}$$

(e) $\int x\sqrt{x-1}\,dx$
Solution:: This is a tricky u-substitution, $u = x-1$. Then, of course $du = dx$, but you also substitute for x as $x = u+1$, so that

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt{u}du \\ &= \int u^{3/2} + \sqrt{u}du \\ &= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c \\ &= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c\end{aligned}$$

(34) Use four rectangles and left endpoints to approximate the area of the region pictured below.



Solution:: The area under the curve would be approximated by that of the left-edge rectangles, whose height is the value of the function at the left endpoint of the subintervals, by:

$$\frac{1}{2}(0) + \frac{1}{2}(1) + \frac{1}{2}(\frac{1}{2}) + \frac{1}{2}(\frac{1}{4})$$

(35) Determine if the function

$$f(x) := \begin{cases} x^2 - 4x + 2, & \text{if } x \geq 3 \\ \frac{\sin(x-3)}{x-3}, & \text{if } x < 3 \end{cases}$$

is continuous at 3.

Solution: This is not continuous, since the left-hand limit

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3} \frac{\sin(x-3)}{x-3} \\ &= \lim_{u \to 0} \frac{\sin(u)}{u} \\ &= 1, \end{aligned}$$

while

$$\begin{aligned} \lim_{x \to 3^+} f(x) &= \lim_{x \to 3} x^2 - 4x + 2 \\ &= 9 - 12 + 2 \\ &= -1, \end{aligned}$$

so the limit at 3 does not exist, and the function cannot be continuous there.

(36) Let $f(x) = \dfrac{\ln x}{x}$

(a) Identify the horizontal asymptotes of this function.

Solution: $y = 0$ (as $x \to +\infty$), because

$$\lim_{x \to +\infty} \frac{\ln(x)}{x} = \lim_{x \to +\infty} \frac{(\frac{1}{x})}{1} = 0.$$

Of course there can't be a horizontal asymptote for $x \to -\infty$ since $f(x)$ is only defined for $x > 0$.

(b) Find an equation of the line tangent to the graph of this function at $x = 2$.

Solution: Since the slope at 2 is $f'(2) = \frac{\frac{1}{2}2 - \ln(2)}{4} = \frac{1-\ln(2)}{4}$, the tangent line at $(2, \frac{\ln(2)}{2})$is

$$\frac{y - \frac{\ln(2)}{2}}{x - 2} = \frac{1 - \ln(2)}{4},$$

or

$$y = \frac{1 - \ln(2)}{4}(x - 2) + \frac{\ln(2)}{2}.$$

(c) Find the area of the region under the graph of this function over the interval $[1, e]$.

Solution: $A = \int_1^e \frac{\ln x}{x}\,dx$, which you integrate by the u-substitution $u = \ln(x)$, $du = \frac{1}{x}dx$, and noting that $u = 1$ when $x = e$ and $u = 0$ when $x = 1$,

$$\begin{aligned} A &= \int_1^e \frac{\ln x}{x}\,dx \\ &= \int_0^1 u\,du \\ &= \frac{1}{2}u^2\Big|_0^1 \\ &= \frac{1}{2}. \end{aligned}$$

(37) A box with a square base and open top must have a volume of 32,000 cubic centimeters. Find the dimensions of the box that minimizes the amount of material used.

Solution: The base of the box is, say, x by x, with height z. The volume is then x^2z and the area of the box (without top) is $x^2 + 4xz$. The constraint is $32000 = x^2z$, so $z = 32000/x^2$, and so the area can be expressed as a function of x by $A(x) = x^2 + 128000/x$. The domain is $0 < x < \infty$, but the limiting values of the area, as x goes either to 0 or ∞, are infinite, so the minimum of A would have to be somewhere in between. The only possibility is a place where $A'(x) = 0$, and

$$A'(x) = 2x - 128000/x^2,$$

so $A'(x) = 0$ implies that $x^3 = 64000$, or $x = 40$. This must be the minimum, since there is only one critical point, and so the dimensions are $40 \times 40 \times 20$, since when $x = 40$, $z = 32000/1600 = 20$.

(38) State the definition of the derivative and use it to compute the derivative of $f(x) := \sqrt{x}$.

Solution: The *definition* of the derivative of a function $f(x)$, as a limit, is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Now, to apply to $f(x) = \sqrt{x}$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{(x+h) - (x)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

(39) A particle moves along the number line with velocity $v(t) = t^2 - t + e^t$ meters per second.

(a) Find a function $s(t)$ which gives the position of this object if its initial position is 3 meters to the right of the origin.

Solution: $s(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 + e^t + 2$, since to have $s(0) = 3$, the constant of integration has to take into account the fact that $e^0 = 1$.

(b) What is the average velocity of the object during the first 3 seconds?

Solution: The average velocity is $\frac{s(3)-s(0)}{3} = \frac{(9-\frac{9}{2}+e^3+2)-3}{3}$.

(c) Find the acceleration $a(t)$.

Solution: $a(t) = v'(t) = 2t - 1 + e^t$.

(40)

(a) State the Mean Value Theorem.

Solution: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a point $c \in (a, b)$ so that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

(b) Use the Mean Value Theorem to explain why, for all a and b, that

$$|\sin(a) - \sin(b)| \leq |a - b|.$$

Solution: For any such a and b (let's assume that $a < b$ for now), since $\sin(x)$ is continuous and differentiable everywhere, there is a $c \in (a, b)$ so that

$$\frac{\sin(b) - \sin(a)}{b - a} = \sin'(c) = \cos(c).$$

But $|\cos(c)| \leq 1$, so (remember, inside absolute values the order of subtraction doesn't matter)

$$\begin{aligned} \left|\frac{\sin(b) - \sin(a)}{b - a}\right| &\leq 1 \\ \frac{|\sin(a) - \sin(b)|}{|a - b|} &\leq 1, \end{aligned}$$

and, multiplying across, $|\sin(a) - \sin(b)| \leq |a - b|$, which is what we wanted to show.

(41) Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.

Solution: Since f is differentiable everywhere, it is also continuous, so we can use MVT to say that there is a $c \in (2, 8)$ with

$$\frac{f(8) - f(2)}{8 - 2} = f'(c).$$

But, since $3 \leq f'(x) \leq 5$ everywhere,

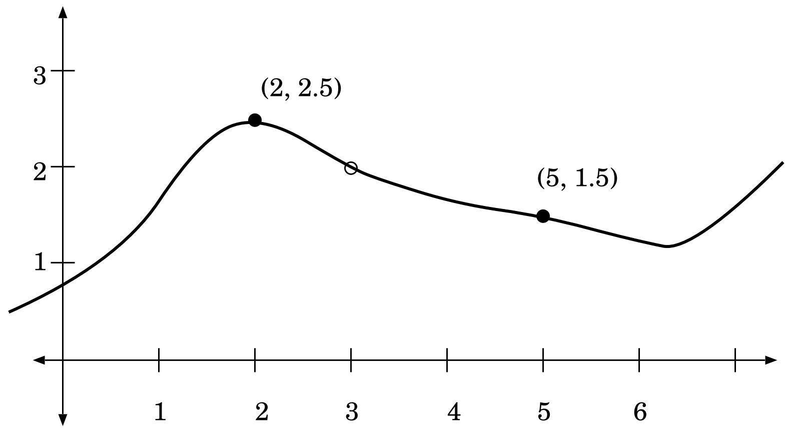
$$\begin{array}{rcl} 3 \leq & f'(c) & \leq 5 \\ 3 \leq & \frac{f(8)-f(2)}{6} & \leq 5 \text{ multiply by 6,} \\ 18 \leq & f(8) - f(2) & \leq 30, \end{array}$$

as was claimed.

(42) Sketch the graph of a function f which is continuous at $x = 1$ but is not differentiable at $x = 1$.

Solution: Take $f(x) = |x - 1|$.

(43) The graph of a function f is as follows:



Find a $\delta > 0$ so that $|f(x) - 2| < .5$ whenever $0 < |x - 3| < \delta$.

Solution: $\delta = 2$ seems to be the largest that will work, since it looks like $f(1) = 1.5$. Going further away from 3 than 2 would involve x's less than 1, whose value is less than 1.5, so would be further away from 2 than 0.5.

(44) Prove using the ϵ, δ definition of limit that $\lim_{x \to 2}(x^2 - 1) = 3$.

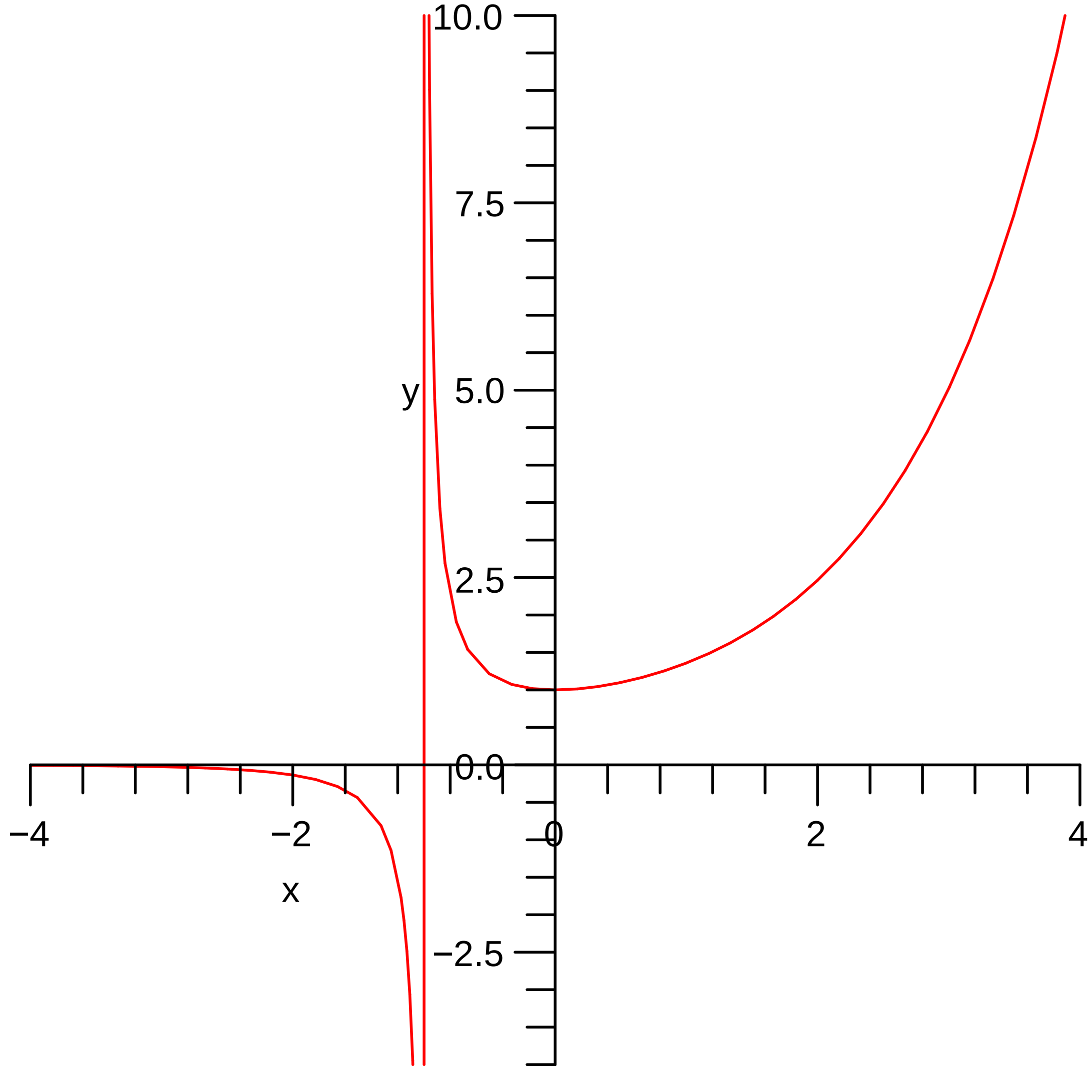
Proof: Let $\epsilon > 0$ be given. Then, if $\delta = \min\{1, \epsilon/5\}$, whenever $0 < |x - 2| < \delta$, we will have $|f(x) - 3| = |(x^2 - 1) - 3| = |x^2 - 4| = |x + 2||x - 2| < 5|x - 2|$ (since $|x - 2| < 1$, $-1 < x - 2 < 1$, or $3 < x + 2 < 5$), and so $|f(x) - 3| < 5|x - 2| < 5(\epsilon/5) = \epsilon$.

(45) Find an overestimate and an underestimate for $\displaystyle\int_0^2 2^{x^2}\,dx$ using four subintervals of equal length.

Solution: The width of the subintervals is $1/2$, and the heights of the inscribed rectangles are the values of the function at the left edges of the subintervals, $x = 0$, $1/2$, 1, and $3/2$, so the underestimate would be $\frac{1}{2}(2^0 + 2^{1/4} + 2^1 + 2^{9/4})$. The heights of the circumscribed rectangles are at the right edges, $x = 1/2$, 1, $3/2$ and 2 so the overestimate would be $\frac{1}{2}(2^{1/4} + 2^1 + 2^{9/4} + 2^4)$.

(46) Sketch the graph of the function $f(x) = \dfrac{e^x}{x+1}$ (including everything).

Solution:: Only one intercept, at $(0, 1)$, and asymptotes at $x = -1$ and $y = 0$ (as $x \to -\infty$ only). $f'(x) = \frac{xe^x}{(x+1)^2}$, so $f' > 0$ when $x > 0$ and $f' < 0$ when $x < 0$, except at $x = -1$, where it is undefined. $f''(x) = \frac{e^x(x^2+1)}{(x+1)^3}$, which is positive for $x > -1$ and negative for $x < -1$. Here, as before, is a computer-generated plot. Due to some bug, there is a vertical line drawn at the vertical asymptote. That is not part of the graph of the function, it is just the asymptote.



(47) State both parts of the Fundamental Theorem of Calculus.

Solution: Part I: If $f(x)$ is continuous on $[a, b]$, then $g(x) := \int_a^x f(t)dt$ satisfies $g'(x) = f(x)$, so is an antiderivative of $f(x)$. Part II: If $F(x)$ is any antiderivative of a continuous function $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

(48) Using the FTC, find $\left(\int_0^{2x} \sin(e^t)\,dt\right)'$. [Oops, that should have been a t inside the integral.]

Solution: If $F(x) = \int_0^x \sin(e^t)dt$, then $F'(x) = \sin(e^x)$ The derivative we need is $\frac{d}{dx}F(2x) = 2F'(2x) = 2\sin(e^{2x})$.

(49) Using only a limit of Riemann sums, evaluate $\displaystyle\int_1^3 x^2\,dx$.

Solution: Split the interval from 1 to 3 into n subintervals, each of width $2/n$. Then using the right-hand edge of the i^{th} subinterval, which is $x_i^* = 1 + 2i/n$, the Riemann sum with this subdivision and these choices for the evaluation point is

$$\begin{aligned}\sum_{i=1}^{n} f(x_i^*)\Delta x &= \sum_{i=1}^{n}(1+2i/n)^2\frac{2}{n}\\ &\sum_{i=1}^{n}(1+\frac{4}{n}i+\frac{4}{n^2}i^2)\frac{2}{n}\\ &= \frac{2}{n}\sum_{i=1}^{n}1+\frac{8}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2\\ &= \frac{2}{n}n+\frac{8}{n^2}\frac{n(n+1)}{2}+\frac{8}{n^3}n(n+1)(2n+1),\end{aligned}$$

by those formulas in the Appendix. The integral is then the limit of these as $n\to\infty$,

$$\begin{aligned}\int_1^3 x^2dx &= \lim_{n\to\infty}\left(\frac{2}{n}n+\frac{8}{n^2}\frac{n(n+1)}{2}+\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}\right)\\ &= 2+4+16/6\\ &= 26/3\end{aligned}$$

(50) Find the area between the curves $y=x^2+2x$ and $y=3x$.
Solution: The curves intersect when $x^2+2x=3x$, or $x=0$ or $x=1$. The area is

$$\begin{aligned}A &= \int_0^1 hdx\\ &= \int_0^1(3x-(x^2+2x))dx\\ &= \int_0^1 x-x^2dx\\ &= \frac{1}{2}x^2-\frac{1}{3}x^3\Big|_0^1\\ &= \frac{1}{6}.\end{aligned}$$

(51) Find the area between the curves $y^2=4-x$ and $3y=x$.
Solution: The two curves intersect when $4-y^2=x=3y$, or $0=y^2+3y-4$, which is when $y=-4$ and $y=1$. The area is only reasonable to find using y as the independent variable. Slice up the region between those two curves by horizontal slices at each $y\in[-4,1]$, of thickness dy and length (from right-hand end to left-hand end) of $(4-y^2)-3y$. The area is the sum of all the areas of these slices, or the integral

$$\begin{aligned}Area &= \int_{-4}^1\left((4-y^2)-3y\right)dy\\ &= 4y-\frac{1}{3}y^3-\frac{3}{2}y^2\Big|_{-4}^1\\ &= \left(4-\frac{1}{3}-\frac{3}{2}\right)-\left(-16+\frac{64}{3}-24\right)\\ &= 20-\frac{65}{3}-\frac{3}{2}+24\\ &= 23-\frac{2}{3}-\frac{3}{2}=\frac{125}{6}=20.833.\end{aligned}$$

I seem to be violating two rules here at the end, one to not simplify, and the other to not use decimal approximations. But in this case at least the first is justified. We are supposed to be finding an area, and it is a good check to see whether the value you have is, indeed, positive. As an even better check, making sure the value is reasonable, not too large or small, is also a good idea. This area has height 5 and width more than 4 at some places (the width is exactly 4 at $y = 0$), so a value of about 20 is within range.

(52) Where is the function $g(x) = \int_0^x \frac{(t+1)(t+2)}{t^4+1} dt$ increasing, and where is it decreasing? What is $g(0)$? Is $g(-1)$ positive or negative? It is true that $\lim_{x\to+\infty} g(x)$ exists (and is finite). You don't need to show that, but, is it positive, or negative? Why?

Solution: Since, by FTC part I, $g'(x) = \dfrac{(x+1)(x+2)}{x^4+1}$, $g'(x) > 0$ for $x > -1$ and $x < -2$, and $g'(x) < 0$ for $-2 < x < -1$. So, $g(x)$ is increasing on $(-\infty, -2]$ and $[-1, \infty)$ and decreasing on $[-2, -1]$.

$g(0) = 0$ since $g(0) = \int_0^0 ***$. Since g is increasing on $[-1, 0]$ and $g(0) = 0$,$g(-1) < 0$. Assuming that $\lim_{x\to+\infty} g(x)$ exists and is finite, since $g(x)$ is increasing on $[0, +\infty)$ and again $g(0) = 0$, $\lim_{x\to+\infty} g(x) > 0$.