MATH 21, FALL, 2008, EXAM # 2

Name:________________________________.

ID #________________________________.

Section #________________________________.

Instructor________________________________.

TA________________________________.

Instructions: All cellphones, calculators, computers, translating devices, and music players must be turned off.

Do all work on the test paper. ***Show all work***. You may receive no credit, even for a correct answer, if no work is shown. You may use the back if you need extra space. ***Do not*** simplify your answers, unless you are explicitly instructed to do so. This does not apply to evaluation of elementary functions at standard values or functional expressions, so that you **would** be expected to simplify $\sin(\pi/6)$ to $\frac{1}{2}$, for example. Do not write answers as decimal approximations; if $\sqrt{2}$ is the answer, leave it that way. *Except where explicitly stated otherwise, you can use the derivative rules and limit rules learned in class.*

You may **not** use a calculator, computer, the assistance of any other students, any notes, crib sheets, or texts during this exam.

You have 60 minutes to complete this exam.

Do not turn to the next page until you are instructed to do so.

Grading:

1.____________/10

2.____________/10

3.____________/5

4.____________/10

5.____________/5

6.____________/10

7.____________/10

8.____________/10

9.____________/10

10.____________/20

Total.____________/100

(1) Find the indicated derivatives: Show the steps involved. *(5 points/part)*

(a) $(\ln(\tan(x)+\sec(x)))'$

SOLUTION: We are asked for the derivative (with respect to x) of $\ln u$, where $u = \tan(x)+\sec(x)$. By a very basic and important application of the Chain Rule, see red box 3 on p. 216, this is (in a different, but equivalent notation) $(\ln\, u)' = \dfrac{u'}{u}$.

Here, we have $u' = \sec^2(x)+\sec(x)\,\tan(x)$, and so:

$(\ln(\tan(x)+\sec(x)))' = \dfrac{\sec^2(x)+\sec(x)\,\tan(x)}{\tan(x)+\sec(x)}$. Getting to here would have been sufficient for full credit, but we indicate a further simplification, both because this WAS literally a Review Sheet Problem (see below) and the further simplifications are given there, but also because the simplified version will prove to be of interest in Mathematics 22, in the context of integration of trigonometric functions.

We also have that:

$\dfrac{\sec^2(x)+\sec(x)\,\tan(x)}{\tan(x)+\sec(x)} = \dfrac{\sec(x)(\sec(x)+\tan(x)}{\tan(x)+\sec(x)} = \sec(x).$

NOTE: This is LITERALLY Problem 1b from the Review Sheet, so see also the Review Sheet solutions. In fact, as we'll see, MANY of the test problems are taken directly from the Review Sheet, and in what follows, we'll simply mention that fact and paste-in the Review Sheet solutions, when that applies. This problem is also quite similar, in spirit, if not in detail, to Practice Problems 3.6.17, 3.6.32 (for differentiating $\ln u$), and to Practice Problems 3.3.11, 3.3.18 (for the derivatives of the trigonometric functions involved in our u).

(b) $\frac{d}{dx}\left(\arcsin(x^2)\right)$

SOLUTION: This is problem 1d from the Review Sheet, except that the arcsin notation is used here. There follows the Review Sheet solution, with the change to the arcsin notation. We should also note that this is quite similar to Practice Problems 3.5.47, 3.5.50, 3.5.53, 3.5.54, all of which combine the derivative of an inverse trigonometric function with the Chain Rule.

Solution:

$$\begin{aligned}\left(\arcsin(x^2)\right)' &= \frac{1}{\sqrt{1-(x^2)^2}}(2x)\\ &= \frac{2x}{\sqrt{1-x^4}}.\end{aligned}$$

(2) Find the indicated limits: Show the steps involved. *(5 points/part)*

(a) $\displaystyle\lim_{x\to 0}\frac{3x-\sin(3x)}{x-x\cos(x)}$

SOLUTION: This is what Problem 39b on the Review Sheet was **intended** to be, so we have to solve this one, not just quote the Review Sheet solution. We note that the limit, as $x \to 0$ of both numerator and denominator is 0, so we are faced with a "$\frac{0}{0}$" form; we try L'Hopital's Rule. I.e., we consider instead:
$\lim_{x\to 0} \frac{(3x - \sin(3x))'}{(x - x\cos(x))'} = \lim_{x\to 0} \frac{3 - 3\cos(3x)}{1 - (\cos(x) - x\sin(x))}$, and now we note that we still have that both numerator and denominator approach 0 as $x \to 0$, i.e., it is STILL a "$\frac{0}{0}$" form; we try L'Hopital's Rule AGAIN. That is, we consider instead:
$\lim_{x\to 0} \frac{(3 - 3\cos(3x))'}{(1 - (\cos(x) - x\sin(x)))'} = \lim_{x\to 0} \frac{9\sin(3x)}{\sin(x) + (\sin(x) + x\cos(x))} = \lim_{x\to 0} \frac{9\sin(3x)}{2\sin(x) + x\cos(x))}$:
STILL a "$\frac{0}{0}$" form; we try L'Hopital's Rule YET AGAIN. That is, we consider instead:
$\lim_{x\to 0} \frac{(9\sin(3x))'}{(2\sin(x) + x\cos(x))'} = \lim_{x\to 0} \frac{27\cos(3x)}{2\cos(x) + \cos(x) - x\sin(x)}$, which, AT LAST, we can find directly: the numerator approaches 27, while the denominator approaches 3, so finally, the limit is 9, and then, by three (count 'em!) applications of L'Hopital's Rule, so is the original limit.
NOTE: This was similar in spirit to several Review Sheet problems that required multiple applications of L'Hopital's Rule, e.g. 39d, 39g. It was also similar in spirit to several Practice Problems on the same theme, e.g. 4.4.21, 4.4.24.

(b) $\lim_{x\to 0^+} (1 + \sin(2x))^{\frac{1}{x}}$

SOLUTION: This is quite similar to Review Sheet Problem 39a except that here, we have $\sin(2x)$ in place of $2x$ (and as $x \to 0$, these are essentially interchangeable, since the limit of their quotient is 1!). What follows is the modification of the Review Sheet solution needed to take into account this change. This was also quite similar to Practice Problems 4.4.54, 4.4.56, 4.4.57, which involved indeterminate powers.

Solution: *This limit is almost the same as one in the notes, and in the book. The idea is to take the logarithm of the function first, so that it is in one of the standard forms (0/0 or ∞/∞), find the limit of that using l'Hôpital's rule, then exponentiate back to find the limit of the original function.*

$$\begin{aligned}
\ln\left((1+\sin(2x))^{1/x}\right) &= \frac{1}{x}\ln(1+\sin(2x)), \textit{ so} \\
\lim_{x\to 0+} \ln\left((1+\sin(2x))^{1/x}\right) &= \lim_{x\to 0+} \frac{\ln(1+\sin(2x))}{x} \; (\textit{of form } 0/0) \\
&= \lim_{x\to 0+} \frac{2\cos(2x)/(1+\sin(2x)}{1} \\
&= 2,
\end{aligned}$$

so

$$\lim_{x\to 0+} (1+\sin(2x))^{1/x} = e^2.$$

(3) Suppose that $f(x)$ and $g(x)$ are differentiable functions so that $f(1) = 1$, $f'(1) = 4$, $f(5) = 3$, $f'(5) = 2$, $f(4) = -3$, $f'(4) = 1$, $g(1) = 5$, $g'(1) = 4$, $g(2) = 1$, and $g'(2) = 3$. Find $\frac{d}{dx}\left[f(g(x))\right]|_{x=1}$. *(5 points)*

SOLUTION: This is literally Review Sheet Problem 3; the solution from there follows. This was also quite similar to Practice Problems 3.4.62, 3.4.63.

Solution: By the chain rule,

$$\begin{aligned} \frac{d}{dx}\left[f(g(x))\right]|_{x=1} &= f'(g(1))g'(1) \\ &= f'(5)\cdot 4 \\ &= 2\cdot 4 = 8. \end{aligned}$$

(4) Show, using the definition of the inverse tangent and implicit differentiation, that

$$(\arctan(x))' = \frac{1}{x^2+1}.$$

(10 points)

SOLUTION: This is literally Review Sheet Problem 7 except that the arctan notation is used here. There follows the Review Sheet solution, with the change to the arctan notation. This was also proved in the textbook, see p. 212, the lines between the two red-boxed formulas, and culminating in the second one.

Solution: If $\theta = \arctan(x)$, then $\tan(\theta) = x$. So, think of a right triangle with opposite side (to the angle θ) x and base 1. It has hypotenuse $\sqrt{1+x^2}$, so $\cos(\theta) = 1/\sqrt{1+x^2}$. But, then, since $\tan(\theta) = x$,

$$\begin{aligned} \frac{d}{dx}(\tan(\theta)) &= 1, \text{ or} \\ \sec^2(\theta)\frac{d\theta}{dx} &= 1. \end{aligned}$$

But then,

$$\begin{aligned} 1 &= \sec^2(\theta)\frac{d\theta}{dx}, \text{ or} \\ \cos^2(\theta) &= \frac{d\theta}{dx} \\ \frac{1}{1+x^2} &= \frac{d\theta}{dx} \\ \frac{1}{1+x^2} &= \frac{d\arctan(x)}{dx}. \end{aligned}$$

(5) Use linear approx or differentials to find an approximate value of $(7.9)^{\frac{1}{3}}$. *(5 points)*

SOLUTION: The statement of the problem provides us with the correct approach; so we take $(7.9)^{\frac{1}{3}}$ to be $f(7.9)$, which we are to approximate as $f(7.9) \approx f(a)+f'(a)(7.9-a)$, for a well-chosen a. Thus, we must identify the general formula for the function f, find its derivative, and then choose a as a value close to 7.9 where we can directly evaluate both $f(a)$ and $f'(a)$. Since $f(7.9) = (7.9)^{\frac{1}{3}}$, it is natural to take $f(x)$ to be $x^{\frac{1}{3}}$, and then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. Now 8 is close to 7.9, and $f(8) = 2$, while $f'(8) = \frac{1}{3}2^{-2} = \frac{1}{12}$, so we take $a = 8$, and then:

$(7.9)^{\frac{1}{3}} \approx 2 + \frac{1}{12}(7.9 - 8) = 2 - \frac{1}{120}.$

NOTE: This was quite similar to Review Sheet Problems 18, 19, and a number of Practice Problems, e.g. 3.10.24, 3.10.25.

(6) MVT

(a) State the Mean Value Theorem. *(5 points)*

SOLUTION: This is literally Review Sheet Problem 28, and the statement of the MVT is pasted in from there:

Solution: If $f(x)$ is a continuous function on $[a, b]$ and is differentiable on (a, b), then there is a point $c \in (a, b)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

(b) Use the Mean Value Theorem to show that the function

$$f(x) := x^3 + 4x - 5$$

only has a root at $x = 1$, and nowhere else. *(5 points)*

SOLUTION: This is literally Review Sheet Problem 29, and while we will paste in the solution from there, it is more of a "Rolle's-ish" solution, so first we give a more directly "MVT-ish" solution. Either sort of solution, correctly formulated, would have been fine. This was also quite similar to Example 2 on p. 281 of the textbook, and to Practice Problems 4.2.17, 4.2.18, 4.2.19.

Note that f, being a polynomial, is differentiable everywhere and in fact, $f'(x) = 3x^2 + 4 \geq 4 > 0$, so, by (a) of the Increasing/Decreasing Test (red box on p. 287), f is increasing everywhere (the interval in question is $(-\infty, \infty)$, the whole real line). Therefore, f is a 1-1 function, and so, for ANY real number r, there is at most one solution to $f(x) = r$. Taking $r = 0$, there is at most one solution of $f(x) = 0$, i.e., at most one root. But by inspection, $x = 1$ IS a root, and so there is a root at $x = 1$ and nowhere else.

Now for the "Rolle's-ish" solution from the Review Sheet solutions:

Solution: If there were two roots of that function, two points a and b at which $f(a) = f(b) = 0$, then by MVT there is a point c between a and b at which

$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$. On the other hand, $f'(x) = 3x^2 + 4$, which is always at least 4, so is never 0. So such a c can't exist, which means that the assumption that there were two roots is false. On the other hand, $f(1) = 0$, so it does indeed have only that one root.

(7) The population of Decatur, Texas grows at a rate proportional to its size. It increased by 60% after 3 years. How long does it take for the population to triple? *(10 points)*

SOLUTION: This is essentially Review Sheet Problem 9 with the following minor changes: the bacteria population has morphed into the population of Decatur, Texas (no inferences were intended so none should be drawn!), 3 days has become 3 years, and we are now interested in how long it takes the population to triple rather than double. The solution from the Review Sheet Solutions is pasted in below, with the necessary modifictions to accomodate these changes. This was also quite similar to Example 1, p. 235 of the textbook, and to Practice Problems 3.8.2, 3.8.3, 3.8.5, 3.8.6.

Solution: Since the population of Decatur grows at a rate proportional to its size, if $P(t)$ is the population, then $P'(t) = kP(t)$, which means that $P(t) = Ae^{kt}$ for some numbers A and k. This is the solution of that general differential equation. To fit this solution to the particular data, we have that $P(0) = A$ (the initial population, which does not need to be a specific number), and $P(3) = 1.6A$, 60% more than we initially had, after 3 years. But then $P(t) = Ae^{kt}$ with the same A as the initial population (at time 0), and

$$\begin{aligned} 1.6A &= P(3) \\ &= Ae^{k3}. \end{aligned}$$

So. $1.6 = e^{3k}$, or, taking natural logs of both sides, $\ln(1.6) = 3k$, or $k = \frac{1}{3}\ln(1.6)$. Then, $P(t)$ is given more explicitly by

$$P(t) = Ae^{\frac{1}{3}\ln(1.6)t}.$$

The time T it takes for the population to triple satisfies

$$\begin{aligned} 3A &= P(T) \\ &= Ae^{\frac{1}{3}\ln(1.6)T}, \end{aligned}$$

so, taking ln() of both sides again, $\ln(3) = \ln(e^{\frac{1}{3}\ln(1.6)T}) = \frac{1}{3}\ln(1.6)T$, so $T = \frac{3\ln(3)}{\ln(1.6)}$.

(8) A 20 foot ladder is leaning against a wall. A painter stands on the top of the ladder, minding his own business. Some fool comes by and ties his dog to the base of the ladder, a cat comes along, and the dog chases after the cat, dragging the base of the ladder with him at a rate of 2 feet per second directly away from the wall. How fast is the painter falling when he is 12 feet from the ground? *(10 points)*

SOLUTION: This is literally Review Sheet Problem 14. Before pasting in the solution from there, it should be noted that even if someone drew the picture with the base of the ladder to the left of the wall, and even if they took x to be the position of the base of the ladder (rather than the distance from the base of the ladder to the base of the wall), with the origin at the base of the wall, so that x is decreasing, rather than increasing, (i.e., dx/dt

is negative), they would also have x < 0, so xdx/dt would STILL be positive. This was also quite similar to Example 2 on p. 242 of the textbook. Now for the paste-in:

Solution: This is a fairly standard related-rates problem. Set x to be the distance from the base of the ladder to the wall, and set y to be the height of the painter above the ground. We **know** that $\frac{dx}{dt} = +2$ and we **want** to know $\frac{dy}{dt}$ **when** $y = 12$. Then, since the ladder has constant length 20 until it smashes against the ground, the Pythagorean theorem tells us that $x^2 + y^2 = 20^2$, which is an equation between what we know and what we want to know. Differentiate both sides of that equation with respect to time to get the relationship between their rates of change, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. At "the when", you have $y = 12$, so plug into the equation $x^2 + y^2 = 20^2$ to see that $x^2 + 12^2 = 20^2$ at that instant, so $x = 16$ (it's a 3-4-5 triangle). Then, you plug these values into the equation relating the rates, and solve for $\frac{dy}{dt}$:

$$\begin{aligned} 0 &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2 \cdot 16 \cdot 2 + 2 \cdot 12 \cdot \frac{dy}{dt}, \end{aligned}$$

or $\frac{dy}{dt} = -\frac{32}{12} = -\frac{8}{3}$, meaning that the painter is falling at 8/3 feet per second at that instant.

(9) Find the absolute maximum and absolute minimum of $f(x) = \sin(x) + \cos(x)$ on $\left[0, \frac{2\pi}{3}\right]$. *(10 points)*

SOLUTION: This is very similar to Problem 25c on the Review Sheet: we have a sum of the basic trigonometric functions, rather than a difference, and closed interval is a little shorter here. A paste-in of the solution from the Review Sheet Solutions follows, with the small modifications necessary to accomodate these minor changes. This was also quite similar to many Practice Problems from Section 4.2, involving finding the absolute max or min of a continuous function on a closed interval, and especially problem 4.1.57, where the function involved is a trigonometric function.

Solution: $f'(x) = \cos(x) - \sin(x)$, so if $f'(x) = 0$, $\sin(x) = \cos(x)$, which happens when $x = \pi/4$ or $x = 5\pi/4$. But only $\pi/4$ is in the interval. $f(0) = 1$ and $f(2\pi/3) = \sqrt{3}/2 - 1/2 = \frac{\sqrt{3}-1}{2} < 1$, and $f(\pi/4) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 1$. So, the max value is $\sqrt{2}$ and the minimum value is $\frac{\sqrt{3}-1}{2}$.

(10) Let $f(x) = \dfrac{x^2 - 1}{x^2 - 9}$. *(20 points)*

SOLUTION: This is very similar to Problem 38 on the Review Sheet. The only difference is in the constants that are subtracted from the quadratic terms. A paste-in of the solution from the Review Sheet Solutions follows, with the small modifications necessary to accomodate these minor changes. Also, a sketch IS supplied in the appended pdf file, while it was not on the Review

Sheet solutions. This problem was also quite similar to Practice Problem 4.3.45.

(a) Find the domain of $f(x)$.

Solution: x can't be ± 3, so the domain is all points except $x = 3$ and $x = -3$.

(b) Find all x- and y- intercepts.

Solution: The y-intercept is $(0, 1/9)$, and the x-intercepts, where $f(x) = 0$, are only at $x = 1$ and $x = -1$.

(c) Find any horizontal or vertical asymptotes.

Solution: There are two vertical asymptotes, $x = 3$ and $x = -3$, where the denominator is 0. There is also one horizontal asymptote, at $y = 1$, because $\lim_{x\to\pm\infty} f(x) = 1$. To be more careful, since $\lim_{x\to 3^+} f(x) = \infty$, $\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to -3^+} f(x) = -\infty$, $\lim_{x\to -3^-} f(x) = \infty$, and, since for x very large in absolute value (but either positive or negative), $f(x) > 1$ (but near 1), so you can find the "tails" of the graph, which will help you sketch the curve. To see the very last point, we write the numerator in terms of the denominator: $x^2 - 1 = (x^2 - 9) + 8$, so
$f(x) = 1 + 8(x^2 - 9)^{-1}$.
This way of viewing f can also help simplify the differentiation and will be exploited in what follows. If x is large in absolute value, then $x^2 - 9 > 0$, and so the term $8(x^2 - 9)^{-1} > 0$, i.e., $f(x) > 1$ (when $|x| > 3$).

(d) Find on what intervals the curve is increasing and decreasing, and find any critical points and local extrema.

SOLUTION: Before giving the (modified version of the) paste-in from the Review Sheet Solution, note how easy the derivative becomes if we start from the rewritten form of f given above:
$f'(x) = -8(x^2 - 9)^{-2} \cdot 2x = -16x(x^2 - 9)^{-2}$. The second factor is positive whenever it is defined, which is whenever $x \neq \pm 3$, and so, f' is defined whenever $x \neq \pm 3$ and has sign opposite to the sign of x. That is, $f'(x) > 0$ on $(-\infty, -3) \cup (-3, 0)$, $f'(0) = 0$, and $f'(x) < 0$ on $(0, 3) \cup (3, \infty)$, and so f is increasing on $(-\infty, -3)$, and on $(-3, 0]$, while f is decreasing on $[0, 3)$ and on $(3, \infty)$. Since $f(\pm 3)$ is undefined, we can't "put together" the two intervals of increase at -3, nor can we do this for the two intervals of decrease at 3. Since $f'(0) = 0$ (and $f(0)$ IS defined), 0 is a critical number. Indeed, $f(0) = 1/9$ is a local max, since f changes from increasing to decreasing at 0. While $f'(\pm 3)$ are undefined, ± 3 are NOT critical numbers, since also f itself is undefined at these values of x. There is no local min.
Now, for the record and for purposes of familiarity/recognition, we supply the modified version of the paste-in:

Solution: Since

$$f'(x) = \frac{2x(x^2 - 9) - 2x(x^2 - 1)}{(x^2 - 9)^2} = \frac{-16x}{(x^2 - 9)^2},$$

so the only critical point is at $x = 0$. The y-value is $1/9$ which is also the y-intercept, of course. For $x > 0$, $f'(x) < 0$ and for $x < 0$, $f'(x) > 0$, except at ± 3 where f' is undefined, so f is decreasing on $[0, 3)$ and on $(3, +\infty)$, and f is increasing on $(-\infty, -3)$ and on $(-3, 0]$.

(e) Find on what intervals the curve is concave up, and where it is concave down, and find any points of inflection.

Solution:

$$\begin{aligned} f''(x) &= -\frac{16(x^2-9)^2 - 64x^2(x^2-9)}{(x^2-9)^4} \\ &= -16\frac{(x^2-9) - 4x^2}{(x^2-9)^3} \\ &= 16\frac{3x^2+9}{(x^2-9)^3}. \end{aligned}$$

The numerator is always positive, so $f''(x) > 0$ if $x > 3$ or $x < -3$, and $f''(x) < 0$ when $-3 < x < 3$. Thus, the graph is CU on $(-\infty, -3) \cup (3, \infty)$ and CD on $(-3, 3)$. While concavity does, therefore, change at $x = \pm 3$, these values of x do not yield inflection points, because, once again, f itself is undefined at these values of x, and so there are no corresponding points on the graph.

(f) Then, sketch the curve, showing each of these features.