MATH 21, FALL, 2008, REVIEW FOR EXAM # 2 SOLUTIONS

The material that will be covered on this exam is from section 3.4 (the chain rule) to section 4.5 (curve sketching), inclusive.

Let Prof. Johnson know (david.johnson@lehigh.edu) if you think you may have found a mistake on any of these solutions. You should look at these solutions ***only*** after you have tried, hard, to solve the problems on your own, without looking at the book or other sources of help. The biggest difficulty during the exam is figuring out what method to apply – how to start working the problem. If you look at the solutions, even without looking through the details, then you will see the method used in the solution, which will make that problem easier to solve, but will not help you learn how to work the problems on your own.

(1) Find the following derivatives

(a) $\dfrac{d}{dx}\left(x^{\sin x}\right)$.

Solution: *First, deal with the expression itself. Always, these expressions with both the base and the exponent functions of x should be converted into exponentials with base e:*

$$\begin{aligned} x^{\sin(x)} &= \left(e^{\ln(x)}\right)^{\sin(x)} \\ &= e^{(\ln(x)\sin(x))}. \end{aligned}$$

Then differentiate:

$$\begin{aligned} \frac{d}{dx}\left(x^{\sin x}\right) &= \frac{d}{dx}\left(e^{(\ln(x)\sin(x))}\right) \\ &= e^{(\ln(x)\sin(x))}\left(\frac{d}{dx}(\ln(x)\sin(x))\right) \\ &= e^{(\ln(x)\sin(x))}\left(\frac{1}{x}\sin(x)+\ln(x)\cos(x)\right). \end{aligned}$$

You could simplify it a bit, but you should not.

(b) $(\ln(\tan(x)+\sec(x)))'$.

Solution: *This has an odd answer:*

$$\begin{aligned} (\ln(\tan x+\sec x))' &= \frac{(\tan x+\sec x)'}{(\tan x+\sec x)} \\ &= \frac{\sec^2(x)+\sec(x)\tan(x)}{(\tan x+\sec x)} \\ &= \frac{(\tan x+\sec x)\sec(x)}{(\tan x+\sec x)} \\ &= \sec(x). \end{aligned}$$

Remember this one; we will need it for integration.

(c) $(\cosh(2x+3))'$.

Solution: Using the chain rule, and the derivative of $\cosh(x)$,

$$\begin{aligned}(\cosh(2x+3))' &= \sinh(2x+3)2 \\ &= 2\sinh(2x+3).\end{aligned}$$

(d) $\frac{d}{dx}\left(\sin^{-1}(x^2)\right)$.

Solution::

$$\begin{aligned}\left(\sin^{-1}(x^2)\right)' &= \frac{1}{\sqrt{1-(x^2)^2}}(2x) \\ &= \frac{2x}{\sqrt{1-x^4}}.\end{aligned}$$

(e) $\left(x^2\ln(x)\right)' =$

Solution: Start with the product rule on this one.

$$\begin{aligned}\left(x^2\ln(x)\right)' &= 2x\ln(x) + x^2\frac{1}{x} \\ &= 2x\ln(x) + x.\end{aligned}$$

(f) $\left(\tan^{-1}(3x+2)\right)'$.

Solution: Again, the chain rule, and the derivative of $\tan^{-1}(x)$ (the inverse tangent).

$$\left(\tan^{-1}(3x+2)\right)' = \frac{3}{1+(3x+2)^2}.$$

(g) $\left((\ln(x)+1)(4x-1)^3\right)'$

Solution: Product rule first, then chain rule.

$$\begin{aligned}\left((\ln(x)+1)(4x-1)^3\right)' &= \frac{1}{x}(4x-1)^3 + (\ln(x)+1)3(4x-1)^2 4 \\ &= \frac{(4x-1)^3}{x} + 12(\ln(x)+1)(4x-1)^2.\end{aligned}$$

(h) $f(x) = \frac{3x^2-2}{x+3}$. Find $f'(x)$.

Solution: This is simply the quotient rule.

$$f'(x) = \frac{(6x)(x+3)-(3x^2-2)1}{(x+3)^2},$$

and of course you stop there, rather than simplify.

(i) Find $\frac{d}{dx}\left((x^2+2)e^{(x^2+3)}\right)$.

Solution: Again use the product rule, and the chain rule on the two pieces,

$$\frac{d}{dx}\left((x^2+2)e^{(x^2+3)}\right) = 2xe^{(x^2+3)} + (x^2+2)e^{(x^2+3)}2x.$$

(j) If $f(x) = \sqrt{\ln(x)}$, find $f'(x)$.

Solution: Chain rule:

$$\begin{aligned}\left(\sqrt{\ln(x)}\right)' &= \frac{1}{2\sqrt{\ln(x)}}\frac{1}{x} \\ &= \frac{1}{2x\sqrt{\ln(x)}}.\end{aligned}$$

(k) $\left(e^{2x}\sin(3x)\right)' =$
Solution:

$$\begin{aligned}\left(e^{2x}\sin(3x)\right)' &= 2e^{2x}\sin(3x) + e^{2x}3\cos(3x)\\ &= 2e^{2x}\sin(3x) + 3e^{2x}\cos(3x).\end{aligned}$$

(l) $\left(\frac{2x-3}{4x+1}\right)' =$
Solution:

$$\begin{aligned}\left(\frac{2x-3}{4x+1}\right)' &= \frac{2(4x+1)-4(2x-3)}{(4x+1)^2}\\ &= \frac{14}{(4x+1)^2}.\end{aligned}$$

(m) $\left((x^2+2x+1)^3(x^2+5x)\right)' =$
Solution:

$$\left((x^2+2x+1)^3(x^2+5x)\right)' = \left(3(x^2+2x+1)^2(2x+2)\right)(x^2+5x) + (x^2+2x+1)^3(2x+5).$$

No need to simplify further.

(n) If $f(x) := \dfrac{\left(x^2-2\right)^2}{x^3+2x+3}$, $f'(x) =$
Solution:

$$\left(\frac{\left(x^2-2\right)^2}{x^3+2x+3}\right)' = \frac{2(x^2-2)^1 2x(x^3+2x+3) - (3x^2+2)(x^2-2)^2}{(x^3+2x+3)^2}.$$

Yes, stop there.

(o) $\frac{d}{dx}\ln(e^x+1)|_{x=0}$
Solution: First, let's find the derivative.

$$\begin{aligned}\frac{d}{dx}\ln(e^x+1) &= \frac{1}{e^x+1}e^x\\ &= \frac{e^x}{e^x+1}.\end{aligned}$$

Then, when $x=0$, $\left.\dfrac{d}{dx}\ln(e^x+1)\right|_{x=0} = \dfrac{e^0}{e^0+1} = \dfrac{1}{2}.$

(p) $\dfrac{d}{dx}e^{\arctan x}|_{x=1}$
Solution: Again, first we will find the derivative, then plug in the value of x.

$$\begin{aligned}\frac{d}{dx}e^{\arctan x} &= e^{\arctan(x)}\frac{1}{x^2+1}\\ &= \frac{e^{\arctan(x)}}{x^2+1}.\end{aligned}$$

Then $\dfrac{d}{dx}e^{\arctan x}|_{x=1} = \dfrac{e^{\arctan(1)}}{1^2+1} = \dfrac{e^{\pi/4}}{2}.$

(q) $\dfrac{d}{dx}\dfrac{\ln|x|}{x}|_{x=-1}$

Solution:

$$\begin{aligned}\frac{d}{dx}\frac{\ln|x|}{x} &= \frac{\frac{1}{x}x - 1\ln|x|}{x^2}\\ &= \frac{1-\ln|x|}{x^2},\end{aligned}$$

so $\frac{d}{dx}\frac{\ln|x|}{x}|_{x=-1} = \frac{1-\ln|-1|}{(-1)^2} = 1.$

(r) $\frac{d}{dx}(6^x)|_{x=2}$

Solution: Here the biggest problem is what to do with the 6. Convert the exponential to base e

$$\begin{aligned}\frac{d}{dx}(6^x) &= \frac{d}{dx}(e^{x\ln(6)})\\ &= \ln(6)e^{x\ln(6)}\\ &= \ln(6)6^x,\end{aligned}$$

so $\frac{d}{dx}(6^x)|_{x=2} = 36\ln(6)$.

(s) $\frac{d}{dx}(\log_3 x)|_{x=7}$

Solution: Again, first convert to base e, then differentiate.

$$\begin{aligned}\frac{d}{dx}(\log_3 x) &= \frac{d}{dx}\left(\frac{\ln(x)}{\ln(3)}\right)\\ &= \frac{1}{x}\frac{1}{\ln(3)}\\ &= \frac{1}{x\ln(3)},\end{aligned}$$

so $\frac{d}{dx}(\log_3 x)|_{x=7} = \frac{1}{7\ln(3)}$.

(2) Suppose that $f(x)$ and $g(x)$ are differentiable functions so that $f(1) = 2$, $f'(1) = 3$, $f(4) = 2$, $f'(4) = 3$, $f(5) = -3$, $f'(5) = 1$, $g(1) = 4$, $g'(1) = 5$, $g(2) = 1$, and $g'(2) = 3$. Find $\frac{d}{dx}\left[f(g(x))\right]|_{x=1}$.

Solution: By the chain rule,

$$\begin{aligned}\frac{d}{dx}\left[f(g(x))\right]|_{x=1} &= f'(g(1))g'(1)\\ &= f'(4)\cdot 5\\ &= 3\cdot 5 = 15.\end{aligned}$$

(3) Suppose that $f(x)$ and $g(x)$ are differentiable functions so that $f(1) = 1$, $f'(1) = 4$, $f(5) = 3$, $f'(5) = 2$, $f(4) = -3$, $f'(4) = 1$, $g(1) = 5$, $g'(1) = 4$, $g(2) = 1$, and $g'(2) = 3$. Find $\frac{d}{dx}\left[f(g(x))\right]|_{x=1}$

Solution: By the chain rule, (Be careful to use the new list of values)

$$\begin{aligned}\frac{d}{dx}\left[f(g(x))\right]\Big|_{x=1} &= f'(g(1))g'(1)\\ &= f'(5)\cdot 4\\ &= 2\cdot 4 = 8.\end{aligned}$$

(4) Assume that $y = f(x)$ satisfies the equation

$$y^5 + 3x^2y^2 + 5x^4 = 12.$$

Find dy/dx in terms of x and y.

Solution: *Here you use implicit differentiation.*

$$\left(y^5 + 3x^2y^2 + 5x^4\right)' = (12)' = 0,$$

so, using the chain rule and remembering that $y = y(x)$ is a function of x,

$$\begin{aligned}0 &= 5y^4\frac{dy}{dx} + 3(2xy^2 + x^2 2y\frac{dy}{dx}) + 20x^3\\ &= \left(5y^4 + 6x^2y\right)\frac{dy}{dx} + 6xy^2 + 20x^3.\end{aligned}$$

Solving for dy/dx gives

$$\frac{dy}{dx} = \frac{-\left(6xy^2 + 20x^3\right)}{5y^4 + 6x^2y}.$$

(5) If $x^2 - 3xy + y^2 = 5$, find $\frac{dy}{dx}$ at the point (1,-1). *(10 points).*

Solution: Again, use implicit differentiation, thinking of y as a function of x:

$$\begin{aligned}(5)' &= \left(x^2 - 3xy + y^2\right)'\\ 0 &= 2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx}\\ &= 2x - 3y - (3x - 2y)\frac{dy}{dx},\end{aligned}$$

so

$$\begin{aligned}(3x - 2y)\frac{dy}{dx} &= 2x - 3y\\ \frac{dy}{dx} &= \frac{2x - 3y}{3x - 2y}.\end{aligned}$$

At the point (1,-1), $\frac{dy}{dx}\Big|_{(1,-1)} = \frac{2\cdot 1 - 3(-1)}{3\cdot 1 - 2(-1)} = 1.$

(6) Find an equation of the tangent line to the graph of $x^3 - xy - y^2 + 5 = 0$ at $(1, 2)$.

Solution: Implicit differentiation, once more. Differentiate both sides:

$$\begin{aligned}0' &= \left(x^3 - xy - y^2 + 5\right)'\\ 0 &= 3x^2 - y - x\frac{dy}{dx} - 2y\frac{dy}{dx} + 0\\ &= 3x^2 - y - \frac{dy}{dx}(x + 2y),\end{aligned}$$

so $\frac{dy}{dx} = \frac{3x^2 - y}{x + 2y}$. At the point $(1, 2)$, $\frac{dy}{dx} = \frac{3-2}{1+4} = \frac{1}{5}$. Now, that finds the slope of the curve at that point. To find the tangent line, plug into the standard point-slope form of the equation of a line:

$$\frac{y-2}{x-1} = \frac{1}{5},$$

or $y = \frac{1}{5}x + \frac{9}{5}$.

(7) Show, using the definition of the inverse tangent and implicit differentiation, that

$$\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}.$$

Solution: If $\theta = \tan^{-1}(x)$, then $\tan(\theta) = x$. So, think of a right triangle with opposite side (to the angle θ) x and base 1. It has hypotenuse $\sqrt{1+x^2}$, so $\cos(\theta) = 1/\sqrt{1+x^2}$. But, then, since $\tan(\theta) = x$,

$$\begin{aligned} \frac{d}{dx}(\tan(\theta)) &= 1, \text{ or} \\ \sec^2(\theta)\frac{d\theta}{dx} &= 1. \end{aligned}$$

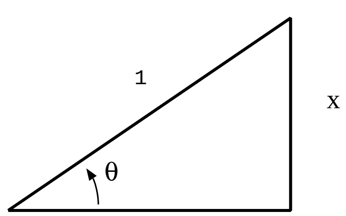
But then,

$$\begin{aligned} 1 &= \sec^2(\theta)\frac{d\theta}{dx}, \text{ or} \\ \cos^2(\theta) &= \frac{d\theta}{dx} \\ \frac{1}{1+x^2} &= \frac{d\theta}{dx} \\ \frac{1}{1+x^2} &= \frac{d\tan^{-1}(x)}{dx}. \end{aligned}$$

(8) *(5 points/part)*

(a) Simplify $\tan(\arcsin(x))$, that is, write that expression without using any trigonometric or inverse trigonometric functions.

Solution:: *I always think of a diagram to figure these out. In this case, the diagram is:*



where $\theta = \arcsin(x)$ *is the angle. You should see that* $\sin(\theta) = x$ *in that drawing. So, the one unlabeled side, the base, has length*

$$\sqrt{1-x^2},$$

so

$$\begin{aligned}\tan(\arcsin(x)) &= \tan(\theta)\\ &= \frac{x}{\sqrt{1-x^2}}.\end{aligned}$$

(b) Use the definition of $\sinh x$ to write $\sinh(\ln(x))$ in terms of algebraic expressions in x.

Solution::

$$\begin{aligned}\sinh(\ln(x)) &:= \frac{1}{2}\left(e^{\ln(x)} - e^{-\ln(x)}\right)\\ &= \frac{1}{2}\left(x - \frac{1}{x}\right).\end{aligned}$$

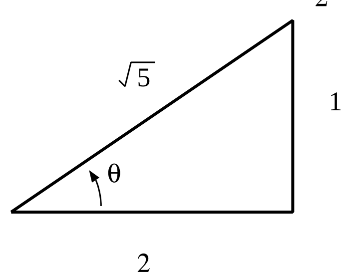
(c) Use the definition of $\cosh x$ to write $\cosh(\ln(x))$ in terms of algebraic expressions in x.

Solution: This is much like the previous part,

$$\begin{aligned}\cosh(\ln(x)) &= \frac{1}{2}\left(e^{\ln(x)} + e^{-\ln(x)}\right)\\ &= \frac{1}{2}\left(x + \frac{1}{x}\right).\end{aligned}$$

(d) Find

$$\sin(\tan^{-1}\frac{1}{2})$$



Solution::

By the picture, $\tan(\theta) = 1/2$, *so*

$$\theta = \tan^{-1}(1/2).$$

Then

$$\begin{aligned}\sin(\tan^{-1}\frac{1}{2}) &= \sin(\theta)\\ &= \frac{1}{\sqrt{5}}.\end{aligned}$$

(9) The population of certain bacteria grows at a rate proportional to its size. It increases by 60% after 3 days. How long does it take for the population to double? (12 points)

Solution: Since the population of the bacteria grows at a rate proportional to its size, if $P(t)$ is the population, then $P'(t) = kP(t)$, which means that $P(t) = Ae^{kt}$ for some numbers A and k. This is the solution of that general differential equation. To fit this solution to the particular data, we have that $P(0) = A$ (the initial population, which does not need to be a specific number), and $P(3) = 1.6A$, 60% more than we initially had, after 3 days. But then $P(t) = Ae^{kt}$ with the same A as the initial population (at time 0), and

$$\begin{aligned} 1.6A &= P(3) \\ &= Ae^{k3}. \end{aligned}$$

So. $1.6 = e^{3k}$, or, taking natural logs of both sides, $\ln(1.6) = 3k$, or $k = \frac{1}{3}\ln(1.6)$. Then, $P(t)$ is given more explicitly by

$$P(t) = Ae^{\frac{1}{3}\ln(1.6)t}.$$

The time T it takes for the population to double satisfies

$$\begin{aligned} 2A &= P(T) \\ &= Ae^{\frac{1}{3}\ln(1.6)T}, \end{aligned}$$

so, taking ln() of both sides again, $\ln(2) = \ln(e^{\frac{1}{3}\ln(1.6)T}) = \frac{1}{3}\ln(1.6)T$, so $T = \frac{3\ln 2}{\ln(1.6)}$.

(10) A new radioactive substance, Doublemintium (D_m), has been found sticking to the undersides of the seats in Packard Auditorium. 70 grams of pure D_m was collected initially. 5 days later only 60 grams of the stuff was still D_m; the rest had decayed into lead and tar. Assuming that the rate of decay of D_m is proportional to the amount present, how much will there be after 20 days?

Solution: Since the amount of D_m present in the sample decays away at a rate proportional to the amount present, if $y(t)$ is the amount of D_m present at time t, then $y'(t) = ky(t)$, which means that $y(t) = Ae^{kt}$ as with a population problem. since 70g was initially found, $A = 70$, and so $y = 70e^{kt}$. But also,

$$\begin{aligned} 60 &= y(5) \\ &= 70e^{k5}, \end{aligned}$$

and so $\frac{6}{7} = e^{5k}$, or $k = \frac{\ln(6/7)}{5}$. Then, the amount present after 20 days will be $y(20)$,

$$\begin{aligned} y(20) &= 70e^{k20} \\ &= 70e^{4\ln(6/7)} \\ &= 70\left(\frac{6}{7}\right)^4. \end{aligned}$$

(11) I bought a cup of coffee at McBurger's. It was far too hot to drink, 90^o Celsius. After 10 minutes, the coffee is at 80^o. The air in McBurger's is kept at an air-conditioned constant of 25^o. How long will I have to wait until the coffee is 70^o and thus cool enough to drink?

Solution: It's best to solve this in terms of the function $y = T - A$, the temperature of the coffee minus the ambient temperature. In that form, since Newton's law of cooling says that the rate of change of temperature of an object is proportional to the difference in temperature between the object and the ambient temperature, then $T' = (T - A)' = y'$ satisfies $y' = ky$, so $y = Be^{kt}$ for some numbers B and k. In this problem $A = 25$. The rest of the problem simply uses the two temperatures at the two times to find the constants B and k. Since $y(0) = 90 - 25 = 65$ is the difference in temperatures between the coffee and the room at time 0, $65 = y(0) = Be^0 = B$. Then, since $55 = y(10)$, $55 = Be^{k10} = 65e^{k10}$, so $\ln(55/65) = k10$, or $k = \frac{\ln(11/13)}{10}$.

Then, to answer the final question, set $T = 70$ and solve for t:

$$\begin{aligned} 70 - 25 &= y(t) \\ 45 &= 65e^{(kt)} \\ &= 65e^{\left(\frac{t\ln(11/13)}{10}\right)}. \end{aligned}$$

So $\ln(9/13) = \frac{t\ln(11/13)}{10}$, or $t = \frac{10\ln(9/13)}{\ln(11/13)}$. As tempting as it may be, you can't cancel those 13's.

(12) A spherical snowball melts in such a way that its volume decreases at the rate of 2 cubic centimeters per minute. At what rate is the radius decreasing when the volume is 400 cubic centimeters?

Solution: The volume of the snowball is $V = \frac{4}{3}\pi r^3$, which is an equation relating the volume to the radius. Differentiate both sides, as functions of t, to see how the rates of change are related: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. You also are given that $\frac{dV}{dt} = 2$. When $V = 400$, then $400 = \frac{4}{3}\pi r^3$, or $r = \left(\frac{300}{\pi}\right)^{1/3}$. At that instant, then $\frac{dV}{dt} = 4\pi\left(\frac{300}{\pi}\right)^{2/3}\frac{dr}{dt}$, so $2 = 4\pi\left(\frac{300}{\pi}\right)^{2/3}\frac{dr}{dt}$, and finally $\frac{dr}{dt} = \frac{2}{4\pi\left(\frac{300}{\pi}\right)^{2/3}} = \frac{1}{2(300)^{2/3}\pi^{1/2}}$.

(13) Egbert is flying his kite. It is 75 feet off the ground, moving horizontally away from Egbert at a rate of 5 feet per second. How fast is Egbert letting out the string when the kite is 100 feet (horizontally) downwind of him?

Solution:: If x is the horizontal distance from Egbert to his kite, and s is the amount of string he has played out, then we know that $dx/dt = +5$, and we want to know ds/dt when $x = 100$. Because of the Pythagorean theorem, $x^2 + 75^2 = s^2$ is the equation relating x to s, which we then differentiate to get a relationship between the rates of change.

$$\begin{aligned} x^2 + 75^2 &= s^2, \text{ so} \\ 2\,x\,\frac{dx}{dt} + 0 &= 2\,s\,\frac{ds}{dt}. \end{aligned}$$

When $x = 100$ $(4 \cdot 25)$, then $s = 125 = 5 \cdot 25$ since the other leg is $75 = 3 \cdot 25$; this is a $3-4-5$ triangle. Plugging in,

$$2\,100\,5 = 2\,125\,\frac{ds}{dt},$$

or

$$\frac{ds}{dt} = \frac{500}{125} = 4.$$

which is the rate at which the line is being played out.

(14) A 20 foot ladder is leaning against a wall. A painter stands on the top of the ladder, minding his own business. Some fool comes by and ties his dog to the base of the ladder, a cat comes along, and the dog chases after the cat, dragging the base of the ladder with him at a rate of 2 feet per second directly away from the wall. How fast is the painter falling when he is 12 feet from the ground?

Solution: This is a fairly standard related-rates problem. Set x to be the distance from the base of the ladder to the wall, and set y to be the height of the painter above the ground. We **know** that $\frac{dx}{dt} = +2$ and we **want** to know $\frac{dy}{dt}$ **when** $y = 12$. Then, since the ladder has constant length 20 until it smashes against the ground, the Pythagorean theorem tells us that $x^2 + y^2 = 20^2$, which is an equation between what we know and what we want to know. Differentiate both sides of that equation with respect to time to get the relationship between their rates of change, $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$. At "the when", you have $y=12$, so plug into the equation $x^2+y^2=20^2$ to see that $x^2+12^2=20^2$ at that instant, so $x=16$ (it's a 3-4-5 triangle). Then, you plug these values into the equation relating the rates, and solve for $\frac{dy}{dt}$:

$$\begin{aligned} 0 &= 2x\frac{dx}{dt}+2y\frac{dy}{dt} \\ &= 2\cdot 16\cdot 2+2\cdot 12\cdot\frac{dy}{dt}, \end{aligned}$$

or $\frac{dy}{dt}=-\frac{32}{12}=-\frac{8}{3}$, meaning that the painter is falling at $8/3$ feet per second at that instant.

(15) Egbert is drinking a daiquiri (non-alcoholic) out of a glass that is a cone with a radius at the top of 3 inches and a height of 5 inches. He drinks his daiquiri at a constant rate of 2 cubic inches per second (through a straw). How fast is the top of the daiquiri falling when it is 4 inches above the bottom of the glass?

Solution: We **know** that the rate of change of the volume V is -2 (in cubic inches per second), $\frac{dV}{dt}=-2$. We **want** to know $\frac{dh}{dt}$, where h is the height of the liquid in his glass, **when** $h=4$.

For this one you need to know what the volume of a cone is. If the base-radius (the base is at the top for this, since in order to hold liquid the glass has to open upwards) is r and the height (from base to apex, which here is pointing downward) is h, the volume is $V=\frac{1}{3}\pi r^2h$. Now, before you think about taking $r=3$ or $h=5$, remember that the glass is only partially full. Even at the time when the daiquiri is 4 inches deep, there is an inch-high gap between the top of the liquid and the top of the glass. So, why mention those numbers? The cone of the entire glass is similar to the cone of the daiquiri in the glass. So, the right triangle formed by the center line from the apex to the top of the liquid to the edge of the glass, then back down to the apex along the glass is a similar triangle to the one formed by the center line from the apex to the top of the glass, then to the edge and back down. The larger triangle has height 5 and base 3, and the smaller has height h and base r. Since the ratios of corresponding sides are the same for similar triangles (or similar cones), $\frac{5}{3}=\frac{h}{r}$, or $r=\frac{3}{5}h$. With that observation, now the volume satisfies $V=\frac{1}{3}\pi\left(\frac{3}{5}h\right)^2h=\frac{3\pi}{25}h^3$. This gives an equation relating the variable V we know something about to the variable h we want to know something about. We then differentiate both sides with respect to time, $\frac{dV}{dt}=\frac{9\pi}{25}h^2\frac{dh}{dt}$, plug in for the fact that $\frac{dV}{dt}=-2$ and (at "the when") $h=4$ [Careful, don't plug in until after you have differentiated], so

$$\begin{aligned} \frac{dV}{dt} &= \frac{9\pi}{25}h^2\frac{dh}{dt} \\ -2 &= \frac{9\pi}{25}4^2\frac{dh}{dt}, \end{aligned}$$

or $\frac{dh}{dt}=-\frac{25}{72\pi}$, the height of liquid in the glass is falling at a rate of $25/(72\pi)$ inches per second.

(16) A man, walking at night, is walking directly towards a streetlight. The light is 10 feet off the ground, and the man is 6 feet tall and walking at 4 feet per second. When the man is 8 feet from the streetlight, how fast is the length of his shadow changing?

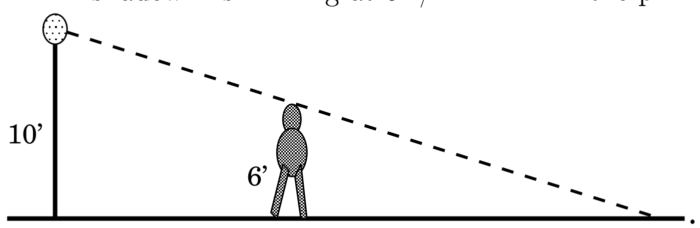
Solution:: Here similar triangles gives the relationship between the variables. Since the big triangle with vertical leg the lightpost is similar to the smaller one (with the man as the vertical leg), we have, if x is the distance from the man to the pole, and y is the length of the shadow, then

$$\frac{6}{y} = \frac{10}{x+y},$$

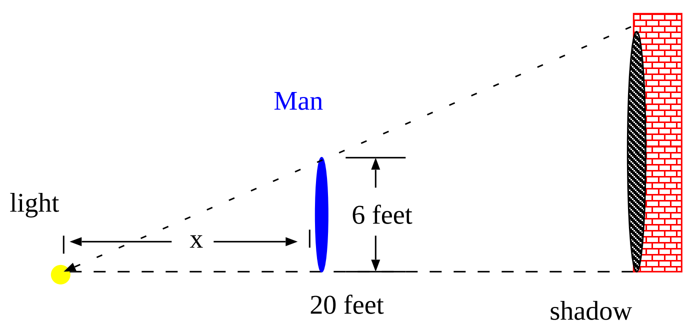
or $6(x+y) = 10y$, or finally, $6x = 4y$. Thus, since you know $\frac{dx}{dt} = -4$, we get immediately that

$$\begin{aligned} 6\frac{dx}{dt} &= 4\frac{dy}{dt}, \text{ or} \\ 6\cdot(-4) &= 4\frac{dy}{dt}, \end{aligned}$$

so $\frac{dy}{dt} = 6$, which doesn't depend upon x so you don't have to plug in $x = 8$ anywhere. The shadow is shrinking at 6ft/sec. Here is the picture



(17) The side of a tall building is illuminated by a floodlight mounted on the lawn in front of the building. The floodlight is at ground level, and is 20 feet from the building. A man, 6 feet tall, is walking towards the floodlight at 4 feet/sec. How fast is his shadow on the wall shrinking when he is 10 feet from the floodlight? **[Correction to the problem: the shadow is actually getting taller, not smaller.]**



Solution: Looking at the picture, the distance labelled x is the distance from the man to the light. Let's also label as y the height of the shadow on the wall. You **know** that $\frac{dx}{dt} = -4$, and you **want** $\frac{dy}{dt}$ **when** $x = 10$. Similar triangles works as for the earlier problem, but a bit differently. The large triangle (light to shadow) has base 20 and height y, but the small triangle has base x and height 6, so similar triangles gives $\frac{y}{20} = \frac{6}{x}$, which is, as we need, an equation relating what we know to what we want. Differentiating both sides of that equation with respect to time gives

$$\frac{1}{20}\frac{dy}{dt} = -\frac{6}{x^2}\frac{dx}{dt},$$

and plugging in $\frac{dx}{dt} = -4$ and (at the time in question) $x = 10$,

$$\begin{aligned} \frac{1}{20}\frac{dy}{dt} &= -\frac{6}{x^2}\frac{dx}{dt} \\ \frac{1}{20}\frac{dy}{dt} &= -\frac{6}{100}(-4), \end{aligned}$$

or $\frac{dy}{dt} = +4.8$. So, to be pedantic about it, the shadow is *shrinking* at -4.8 feet/sec, or, better phrased, the shadow is growing at 4.8 ft/sec.

(18) Use differentials to approximate

$$\sqrt{24}.$$

Solution:

$$\sqrt{24} = f(24)$$

where

$$f(x) = \sqrt{x}.$$

Now, linear approximation says that $f(x) \approx f(a) + f'(a)(x-a)$. Take $a = 25$ since it is easy to evaluate square roots there, and then $\Delta x = -1$, or

$$f(x) = \sqrt{x} \approx 5 + \frac{1}{2 \cdot 5}(x - 25).$$

Taking $x = 24$ gives

$$f(24) = \sqrt{24} \approx 5 - \frac{1}{10} = 4.9.$$

(19) Use differentials to approximate $\sqrt{97}$. *(10 points).*

Solution: Again take $f(x) = \sqrt{x}$, but this time take $a = 100$, the nearest point with a "good" value for $f(x)$. Linear approximation, or differentials, gives that $f(x) \approx f(a) + f'(a)(x-a)$, or, if $df = f(x) - f(a)$ is the amount f changes, and $dx = x - a$ is the amount x changes, $df \approx f'(a)dx$. Now, for this value, $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(a) = f'(100) = \frac{1}{20}$, $f(a) = 10$, and so

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ \sqrt{x} &\approx 10 + \frac{1}{20}(x - 100). \end{aligned}$$

Since $x = 97$, or $dx = -3$,

$$\begin{aligned} \sqrt{97} &\approx 10 + \frac{1}{20}(97 - 100) \\ &\approx 10 - \frac{3}{20} \\ &\approx \frac{197}{20}. \end{aligned}$$

(20) If the radius of a circle is measured to be 12 inches, with an error of $\pm 1/8$ inch, then how much error might there be in the calculation of the area of the circle? (Use differentials). *(10 points)*

Solution: The function is the area of the circle, $f(r) = \pi r^2$, where r is the radius. The error is the maximum that $df = f(r) - f(a)$ could be, where a is the measured distance (12 inches) and r is the real distance, which is somewhere between $12\frac{1}{8}$ inches and $12 - \frac{1}{8}$ inches, so dr is at most $\frac{1}{8}$. Using differentials, with $a = 12$,

$f(r) - f(a) = df \approx f'(a)dr$ becomes $f(r) \approx 2\pi a\, dr = 24\pi\, dr$ which would be at most $24\pi\frac{1}{8} = 3\pi$, so the maximum error in the area is 3π square inches.

(21) Approximate $\sqrt[3]{7.5}$ using differentials.

Solution: Here the function $f(x) = \sqrt[3]{x}$, and so we choose $a = 8$. Note that $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. Using the linear approximation formula, $f(x) \approx f(a) + f'(a)(x-a)$, we have

$$\begin{aligned} f(x) &= \sqrt[3]{x} \\ &\approx f(a) + f'(a)(x-a) \\ &\approx \sqrt[3]{8} + \frac{1}{12}(x-8), \end{aligned}$$

so $\sqrt[3]{7.5} \approx 2 + \frac{1}{12}(-\frac{1}{2}) = 47/24$.

(22) Find the critical numbers (critical points) of the function $f(x) = 2x^3 + 3x^2 - 12x + 1$.

Solution::

The function is differentiable everywhere, so look at the derivative and find out where it is 0 in the domain.

$$\begin{aligned} f'(x) &= 6x^2 + 6x - 12 \\ &= 6(x^2 + x - 2) \\ &= 6(x-1)(x+2), \end{aligned}$$

which is 0 *at* 1 *and* -2 *only. So, the function has critical points* $x = 1$ *and* $x = -2$*, with critical values, respectively,* -6 *and* 21*.* $x = 1$ *is a local minimum, and* $x = -2$ *is a local maximum, though you do not have to mention that here.*

(23) For the function

$$f(x) = x^2 - 3x + 1,$$

defined on the interval $[0, 2]$, find the maximum value and where it occurs. *(10 points)*

Solution:

$$f'(x) = 2x - 3$$

which is 0 only when $x = 3/2$*. So, the maximum value occurs either at* $x = 0$*,* $x = 3/2$*, or* $x = 2$*.*

$$\begin{aligned} f(0) &= 1 \\ f(3/2) &= \frac{9}{4} - \frac{9}{2} + 1 \\ &= -\frac{5}{4}, \textit{ and} \\ f(2) &= -1, \end{aligned}$$

so the maximum value is 1, which occurs at $x = 0$*.*

(24) Find the critical numbers of the function $F(x) = x^{4/5}(x-4)^2$.

Solution: Since

$$\begin{aligned} F'(x) &= \frac{4}{5}x^{-\frac{1}{5}}(x-4)^2 + 2x^{\frac{4}{5}}(x-4) \\ &= \frac{4(x-4)^2 + 10x(x-4)}{5x^{\frac{1}{5}}} \\ &= \frac{2(x-4)(7x-8)}{5x^{\frac{1}{5}}}, \end{aligned}$$

then there are critical points (AKA critical numbers) at $x = 4$, $x = 8/7$ where the derivative vanishes, and $x = 0$ where the derivative does not exist. You do have to include $x = 0$ on this one. $x = 4$ is a local minimum, $x = 8/7$ is a local maximum, which you can see by the first derivative test (or the second derivative test, if you wanted to compute F''). $x = 0$ is also a local minimum, since for $x > 0$ (but near 0), $F'(x) > 0$, and for $x < 0$, $F'(x) < 0$. Again by the first (not second) derivative test, that is a local minimum point.

(25) Without using a calculator, find the absolute maximum and absolute minimum values of $f(x)$ on the given interval.

(a) $f(x) = x^2 - 6x + 10$, $[2, 5]$

Solution: $f'(x) = 2x - 6$, which is zero only when $x = 3$. So, the possible places where the maximum or minimum might be are at $x = 2, 3$, or 5. $f(2) = 2$, $f(3) = 1$, and $f(5) = 5$, so the maximum value is 5 and the minimum value is 1.

(b) $f(x) = (x + 1)/\sqrt{x^2 + 1}$, $[0, 3]$

Solution:

$$\begin{aligned} f'(x) &= \frac{1\sqrt{x^2+1} - (x+1)\frac{x}{\sqrt{x^2+1}}}{(x^2+1)} \\ &= \frac{\left(x^2+1\right) - (x+1)x}{(x^2+1)^{3/2}} \\ &= \frac{1-x}{(x^2+1)^{3/2}}, \end{aligned}$$

so $f'(x) = 0$ only when $x = 1$. So, we compare $f(0) = 1$, $f(1) = \sqrt{2}$, and $f(3) = \frac{4}{\sqrt{10}}$. Can we do this without a calculator? Of course we can; $3 < \sqrt{10} < 4$, so $1 < f(3) < 4/3$, and $\sqrt{2}$ is about 1.4, so is bigger than 4/3. Thus the minimum is 1 and the maximum is $\sqrt{2}$.

(c) $f(x) = \sin x - \cos x$, $[0, \pi]$

Solution: $f'(x) = \cos(x) + \sin(x)$, so if $f'(x) = 0$, $\sin(x) = -\cos(x)$, which happens when $x = 3\pi/4$ or $x = 7\pi/4$. But only $3\pi/4$ is in the interval. $f(0) = -1$ and $f(\pi) = +1$, and $f(3\pi/4) = +\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$. So, the max value is $\sqrt{2}$ and the minimum value is -1.

(d) $f(x) = (x - 2)\ln(x - 2)$, $[\frac{7}{3}, 3]$

Solution: $f'(x) = 1\ln(x - 2) + 1$, so $f'(x) = 0$ only when $\ln(x - 2) = -1$, or $x - 2 = \frac{1}{e}$, $x = \frac{1}{e} + 2$. Wait, is this in the interval $[\frac{7}{3}, 3]$? Since $2 < e < 3$, $\frac{1}{3} < \frac{1}{e} < \frac{1}{2}$, so $\frac{1}{e} + 2 \in [\frac{7}{3}, 3]$. Now evaluate: $f(7/3) = \frac{1}{3}\ln(1/3) < 0$, $f(\frac{1}{e} + 2) = \frac{1}{e}\ln(1/e) = -\frac{1}{e} < 0$, and $f(3) = 0$. So, certainly 0 is the maximum value. Which of the others is smaller? Without using a calculator. Well, you know that $x = \frac{1}{e} + 2$ is a critical point, and $f''(x) = \frac{1}{x-2} > 0$ in this interval, so that interior critical point is a local minimum. But, for any function continuous on $[a, b]$ which is differentiable on (a, b) and has only one critical point on (a, b), then if that interior critical point is a local minimum, it is an absolute minimum (if there were a lower point there would have to be another zero of the derivative, by Rolle's theorem), so we know that the interior point is the absolute minimum, or the minimum value is $-\frac{1}{e}$.

(e) $f(x) = x\sqrt{e} - e^x + 2$, $[0, 1]$

Solution: The derivative is $f'(x) = \sqrt{e} - e^x$. Careful, the derivative of $\sqrt{e}$ is 0. It is a constant. Do not use the product rule. So, $f'(x) = 0$ only when $x = 1/2$. The three points are $0, 1/2$, and 1. $f(0) = 1$, $f(\frac{1}{2}) = 2 - \frac{\sqrt{e}}{2}$, and $f(1) = \sqrt{e} - e + 2$. Again, it is not simple to see where the maximum and minimum are, but $1 < \sqrt{e} < 2$ and $2 < e < 3$, so $f(1/2) > 1 = f(0)$. Now, how do you compare the last two? As with the previous part, $x = \frac{1}{2}$ is a local maximum by the second derivative test, because $f''(x) = -e^x < 0$. Since it is a local maximum, and is the only interior critical point, it must be an absolute maximum. So, the maximum value is $2 - \frac{\sqrt{e}}{2}$. To figure out which is the minimum, you can notice that $\frac{3}{2} < \sqrt{e}$, because $\frac{9}{4} = 2.25 < e$. You also need to remember that $e > 2.5$ (it's 2.718281828....), so $f(1) = \sqrt{e} - e + 2 > \frac{3}{2} - 2.5 + 2 = 1 = f(0)$, and so 1 is the minimum value.

(26) Find the critical points of the function $f(x) = 4x^3 + 3x^2 - 6x + 2$.

Solution: The only critical points of this function are where $f'(x) = 0$, and

$$\begin{aligned} f'(x) &= 12x^2 + 6x - 6 \\ &= 6(2x^2 + x - 1) \\ &= 6(2x - 1)(x + 1), \end{aligned}$$

which you can factor by guessing. So, the only critical points are at $x = 1/2$ and $x = -1$. By either the first or second derivative test, since $f''(x) = 24x + 6$, $x = 1/2$ is a local minimum and $x = -1$ is a local maximum, but you didn't need to show that.

(27) Find the maximum value of the function

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1 \\ x^2 - 3x + 3, & \text{if } 1 < x \le 2 \end{cases}$$

on the interval $[0, 2]$.

Solution: Here, you really should point out that $f(x)$ is continuous on $[0, 2]$, since the two pieces of the graph match up. So, there is a maximum value. You wouldn't want to waste energy looking for something that doesn't exist. It can only occur at an endpoint, a place where the derivative vanishes, or $x = 1$, where the derivative does not exist.

$$f'(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 2x - 3, & \text{if } 1 < x < 2 \end{cases},$$

so $f'(x) = 0$ only at $x = 3/2$. Comparing, $f(0) = 0$, $f(1) = 1$, $f(3/2) = \frac{9}{4} - \frac{9}{2} + 3 = \frac{3}{4}$, and $f(2) = 4 - 6 + 3 = 1$. So, the maximum value is 1, which occurs at two different points, $x = 1$ and $x = 2$.

(28) State the Mean Value Theorem. *(5 points)*

Solution: If $f(x)$ is a continuous function on $[a, b]$ and is differentiable on (a, b), then there is a point $c \in (a, b)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

(29) Use the Mean Value Theorem to show that the function

$$f(x) := x^3 + 4x - 5$$

only has a root at $x = 1$, and nowhere else. *(5 points)*

Solution: If there were two roots of that function, two points a and b at which $f(a) = f(b) = 0$, then by MVT there is a point c between a and b at which $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$. On the other hand, $f'(x) = 3x^2 + 4$, which is always at least 4, so is never 0. So such a c can't exist, which means that the assumption that there were two roots is false. On the other hand, $f(1) = 0$, so it does indeed have only that one root.

(30) Verify that the function $f(x) = x + \ln x$ satisfies the hypotheses of the Mean Value Theorem on the interval $1 \leq x \leq e$ and find all numbers c which satisfy the conclusion of the Mean Value Theorem.

Solution: f is differentiable on $(0, \infty)$, so is both differentiable and continuous on $[1, e]$. MVT guarantees that there is a point $c \in (1.e)$ at which

$$\frac{f(e) - f(1)}{e - 1} = f'(c),$$

but doesn't say for how many c this will hold. In this case, $f'(c) = 1 + \frac{1}{c}$, and so this equation holds when

$$\begin{aligned} 1 + \frac{1}{c} &= \frac{f(e) - f(1)}{e - 1} \\ &= \frac{(e+1) - 1}{e - 1} \\ &= \frac{e}{e - 1}, \end{aligned}$$

so

$$\begin{aligned} e &= (e-1)(1 + \frac{1}{c}) \\ &= e + \frac{e}{c} - 1 - \frac{1}{c}, \text{ or} \\ 1 &= \frac{e-1}{c}, \end{aligned}$$

or $c = e - 1$. There is, in this case, only one c that works. We know that there had to be at least one solution from the MVT, but it does not say exactly where.

(31) Suppose $f(x)$ is differentiable every every real number x and $|f'(x)| \leq 7$ for all x. Show, using the Mean Value Theorem, that

$$|f(x_2) - f(x_1)| \leq 7|x_2 - x_1|$$

for all x_1, x_2. *(12 points)*

Solution:: *By the MVT,*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

for some c between those two points. But, $|f'(x)| \leq 7$, *so*

$$\begin{aligned} \left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right| &= \left|f'(c)\right| \\ \frac{|f(x_2)-f(x_1)|}{|x_2-x_1|} &\leq 7. \end{aligned}$$

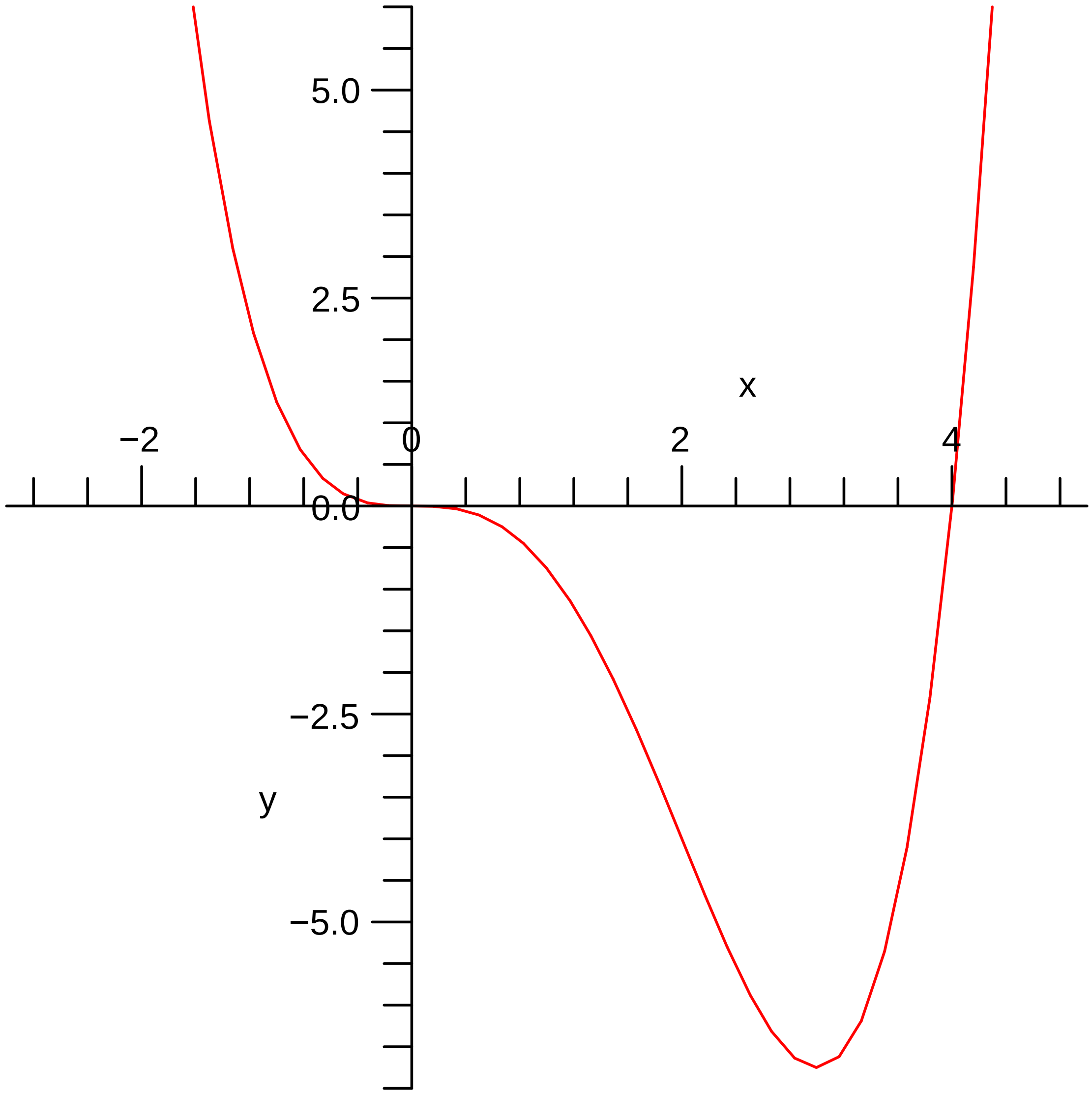
Multiplying both sides by the denominator on the left yields the result.

(32) If all you know about a (differentiable) function $f(x)$ is that its derivative $f'(x)$ is zero for only two values of x, show that the equation $f(x) = 3$ can have no more than three solutions. *(10 points)*

Solution: This is a contradiction argument. Note that we have to know that $f'(x)$ exists for all x, since it's nonzero for all but two values. So f is differentiable at all x, and so also continuous everywhere. Thus, Rolle's theorem and the MVT both can be applied. If there were more than three points where $f(x) = 3$, then there are, what, at least 4, x_1, x_2, x_3, and x_4. We might as well assume we labeled them in order, $x_1 < x_2 < x_3 < x_4$. Rolle's theorem (or, of course, the full MVT) will imply that there is a point $c_1$between $x_1$and $x_2$so that $f'(c_1) = 0$, and there is a point $c_2$between $x_2$and x_3 where $f'(c_2) = 0$, and there is a point c_3 between x_3 and x_4 where $f'(c_3) = 0$. But c_1, $c_2$and c_3 are on different, nonintersecting intervals, so they really have to be different points. That is too many points, since $f'(x)$ can only be 0 at most twice. That means these 4 points can't exist. So, there can't be more than 3 solutions of $f(x) = 3$.

(33) Sketch the graph of the function $f(x) = \frac{1}{4}x^4 - x^3$. Include (with labels) all local extreme points, all absolute extreme points, where the curve is increasing or decreasing, all inflection points and concavity, and all intercepts.

Solution: We start with the domain, which is clearly the entire line. We then find the intercepts, which are only at $(0, 0)$ and $(4, 0)$. There are no asymptotes or symmetry. We then look at the derivatives: $f'(x) = x^3 - 3x^2 = x^2(x-3)$, so f has critical points at $x = 0, 3$, with values $f(0) = 0$ and $f(3) = -27/4$. $f'(x) > 0$ for $x > 3$, and $f' < 0$ for $x < 0$ or $0 < x < 3$, so it levels off only at $x = 0$, but has a local minimum at $x = 3$. For the second derivative, $f'' = 3x^2 - 6x = 3x(x-2)$, which gives inflection points when $x = 0, 2$, so the inflection points are $(0, 0)$ and $(2, -4)$, and for $x > 2$ or $x < 0$, $f''(x) > 0$, but if $0 < x < 2$, $f''(x) < 0$. Here is the drawing (which I cheated on and generated with Maple):



(34) Sketch the graph of the function $f(x) = 3(x^2 - 1)/(x + 2)$, including (with labels) all intercepts, asymptotes (including slant-asymptotes), local extreme points, and where the curve is increasing and decreasing.

Solution: Intercepts: (0,-3/2), (1,0) and (-1,0). Asymptotes: $x = -2$, (going to $+\infty$ from the right, and $-\infty$ from the left) and a slant-asymptote of slope 3 because the ratio of the highest-order terms of the numerator and denominator are $3x$, but, since

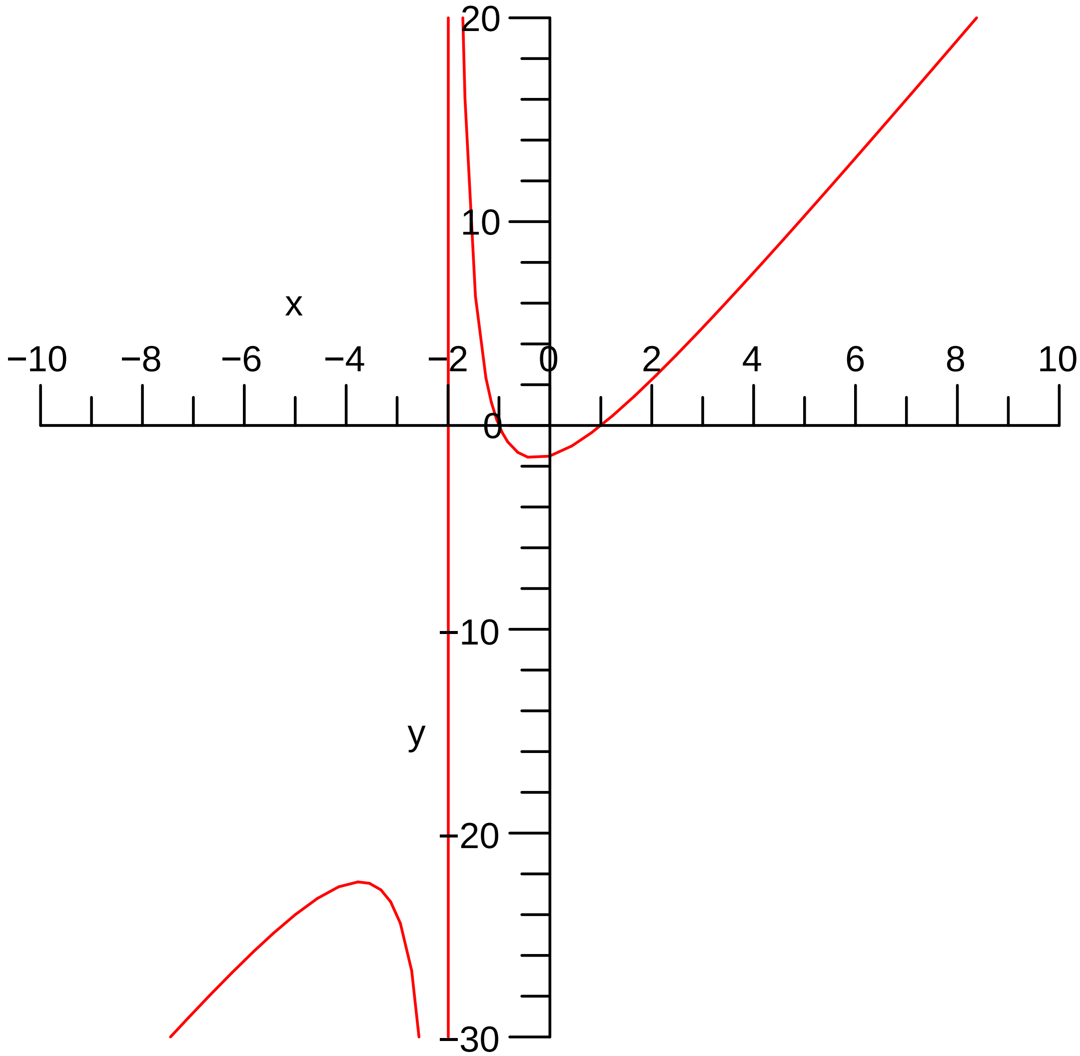
$$\begin{aligned}
\lim_{x\to\infty} f(x) - 3x &= \lim_{x\to\infty} \frac{3(x^2-1)}{x+2} - 3x \\
&= \lim_{x\to\infty} \frac{3(x^2-1) - 3x(x+2)}{x+2} \\
&= \lim_{x\to\infty} \frac{-6x-3}{x+2} \\
&= -6,
\end{aligned}$$

so $y = 3x - 6$ is the slant-asymptote, since

$$\begin{aligned}
\lim_{x\to\infty} f(x) - (3x-6) &= \lim_{x\to\infty} \frac{3(x^2-1)}{x+2} - (3x-6) \\
&= \lim_{x\to\infty} \frac{3(x^2-1) - (3x-6)(x+2)}{x+2} \\
&= \lim_{x\to\infty} \frac{+9}{x+2} \\
&= 0.
\end{aligned}$$

To be subtle about it, you can see that, as you approach $+\infty$, $f(x)$ is above the slant-asymptote, and as you approach $-\infty$, $f(x)$ is below the slant-asymptote, since $f(x) - (3x-6)$ is respectively positive or negative.

Beyond this, we only have to look at the first derivative. $f'(x) = \frac{3(x^2+4x+1)}{(x+2)^2}$, which is 0 when $x = \frac{-4\pm\sqrt{12}}{2} = -2\pm\sqrt{3}$, with values at the critical points $(-2-\sqrt{3}, -2\sqrt{3}(3+2\sqrt{3}))$, $(-2+\sqrt{3}, -2\sqrt{3}(-3+2\sqrt{3}))$. f' is negative between these two critical points, and positive elsewhere, except of course at $x = -2$, where it is infinite. Here is the plot, again generated by Maple:



(35) Sketch the graph of $f(x) = 2x^3 - 3x^2 + 1$, showing all intercepts, asymptotes, where the curve is increasing or decreasing, any critical points, concavity, and any points of inflection. As a hint, $f(1) = 0$. *(15 points)*

Solution: *The* y*-intercept is* $(0, 1)$*, and the* x*-intercepts are solutions of*

$$0 = 2x^3 - 3x^2 + 1.$$

The hint helps immediately,

$$\begin{aligned} 0 &= 2x^3 - 3x^2 + 1 \\ &= (x-1)?? \\ &= (x-1)(2x^2 - x - 1) \\ &= (x-1)(2x+1)(x-1) \\ &= (x-1)^2(2x+1), \end{aligned}$$

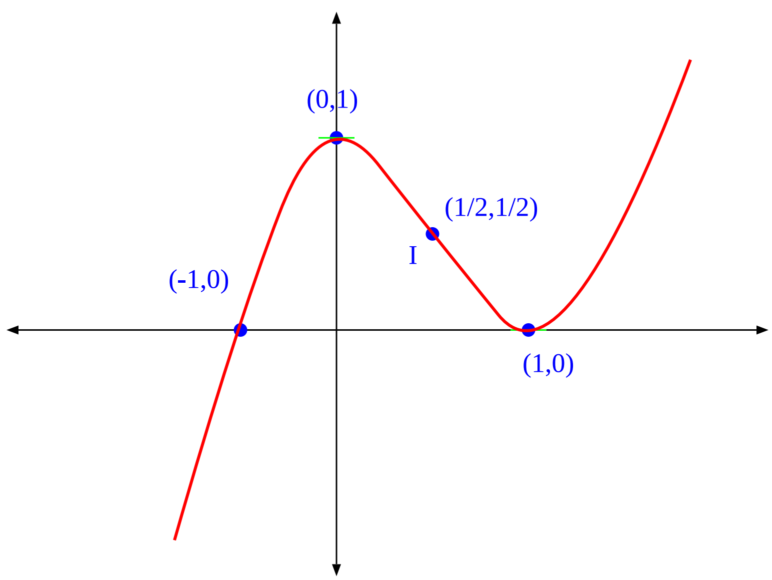
so it has roots at only $x = 1$ and $x = -1/2$. Thus, the x-intercepts are $(1, 0)$ and $(-1/2, 0)$.
There are no asymptotes, since the function is defined everywhere and does not have a limit as x goes to infinity.
To see the slope, take the derivative.

$$\begin{aligned} f'(x) &= 6x^2 - 6x \\ &= 6x(x-1), \end{aligned}$$

which has roots at $x = 0$ and $x = 1$ only, so critical points are $(0, 1)$ and $(1, 0)$. On $(-\infty, 0]$ the curve is increasing, on $[0, 1]$ decreasing, and increasing again on $[1, \infty)$.
For concavity, differentiate again.

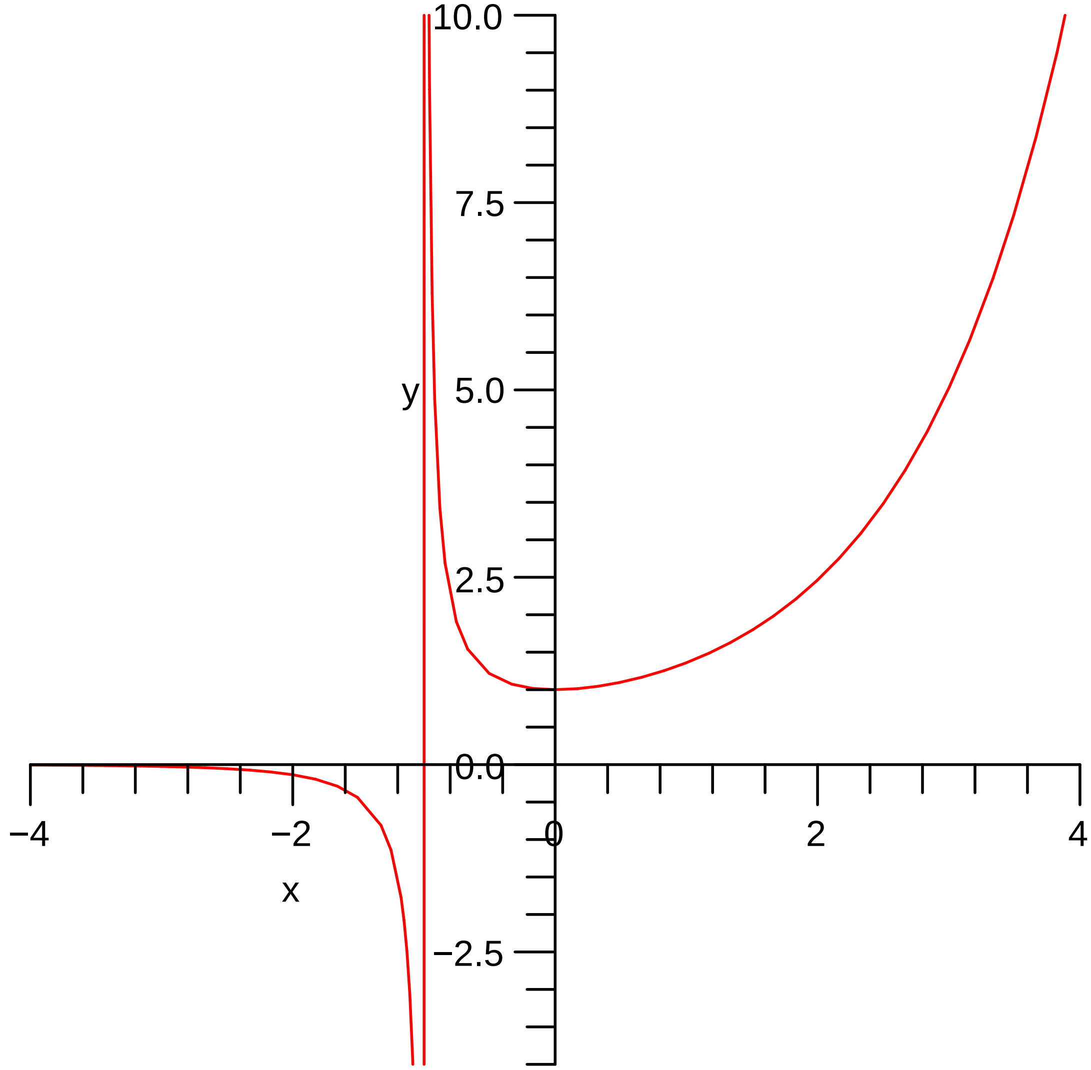
$$f''(x) = 12x - 6 = 6(2x - 1),$$

which has a root only at $x = 1/2$. Thus there is only one inflection point, at $(1/2, 1/2)$ ($f(1/2) = 1/2$). For $x < 1/2$ the curve is concave down, and for $x > 1/2$ it is concave up. Here is a drawing showing all those points.



(36) Sketch the graph of the function $f(x) = \dfrac{e^x}{x+1}$ (including everything).

Solution: Only one intercept, at $(0, 1)$, and asymptotes at $x = -1$ and $y = 0$ (as $x \to -\infty$ only). $f'(x) = \frac{xe^x}{(x+1)^2}$, so $f' > 0$ when $x > 0$ and $f' < 0$ when $x < 0$, except at $x = -1$, where it is undefined. $f''(x) = \frac{e^x(x^2+1)}{(x+1)^3}$, which is positive for $x > -1$ and negative for $x < -1$. Here, as before, is a computer-generated plot:



(37) Below is the graph of a function f defined on the open interval $0 < x < 4$. Find

(a) The critical numbers

Solution: At $x = 1$ there is a 0 derivative, and at $x = 2$ there is no derivative. Both are critical points (or critical "numbers" – meaning just the x). The endpoints are not part of the domain, so these would be the only critical points. Because $x = 0$ and $x = 4$ are specifically excluded, they cannot be critical points, nor can they be local extrema or anything else.

(b) The local extrema

Solution: $x = 1$is a local minimum, and $x = 2$ is a local maximum.

(c) The absolute extrema

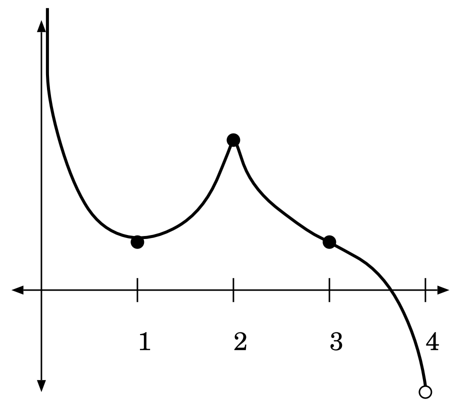
Solution: There is neither an absolute max nor min. Those points would have to be the endpoints, but as I have been saying, they are not there.

(d) Where the graph is concave up and where concave down.

Solution: Concave up for $0 < x < 2$, and $2 < x < 3$, concave down for $x > 3$.

(e) The inflection points.

Solution: $x = 3$ only.



(38) Let $f(x) = \frac{x^2-4}{x^2-1}$, then

(a) Find the domain of $f(x)$.

Solution: x can't be ± 1, so the domain is all points except $x = 1$ and $x = -1$.

(b) Find all x- and y- intercepts.

Solution: The y-intercept is $(0, 4)$, and the x-intercepts, where $f(x) = 0$, are only at $x = 2$ and $x = -2$.

(c) Find any horizontal or vertical asymptotes.

Solution: There are two vertical asymptotes, $x = 1$ and $x = -1$, where the denominator is 0. There is also one horizontal asymptote, at $y = 1$, because $\lim_{x\to\pm\infty} f(x) = 1$. To be more careful, since $\lim_{x\to 1^+} f(x) = -\infty$, $\lim_{x\to 1^-} f(x) = \infty$, $\lim_{x\to -1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = -\infty$, and, since for x very large in absolute value (but either positive or negative), $f(x) < 1$ (but near 1), so you can find the "tails" of the graph, which will help you sketch the curve.

(d) Find where the curve is increasing and decreasing, and find any critical points.

Solution: Since

$$\begin{aligned} f'(x) &= \frac{2x(x^2-1) - 2x(x^2-4)}{(x^2-1)^2} \\ &= \frac{6x}{(x^2-1)^2}, \end{aligned}$$

so the only critical point is at $x = 0$. The y-value is 4 which is also the y-intercept, of course. For $x > 0$, $f'(x) > 0$ and for $x < 0$, $f'(x) < 0$, except at ± 1 where f' is undefined, so f is increasing on $[0, 1)$ and on $(1, +\infty)$, and f is decreasing on $(-\infty, -1)$ and on $(-1, 0]$.

(e) Find on what regions the curve is concave up, and where it is concave down, and find any points of inflection.

Solution:

$$\begin{aligned} f''(x) &= \frac{6(x^2-1)^2 - 24x^2(x^2-1)}{(x^2-1)^4} \\ &= \frac{6(x^2-1) - 24x^2}{(x^2-1)^3} \\ &= \frac{-18x^2-6}{(x^2-1)^3}. \end{aligned}$$

The numerator is always negative, so $f''(x) < 0$ if $x > 1$ or $x < -1$, and $f''(x) > 0$ when $-1 < x < 1$.

(f) Then, sketch the curve, showing each of these features. *(20 points)*

Solution: Put all this information together, and draw a sketch showing all these features.

(39) Find the following limits: *(5 points/part)*

(a) $\lim\limits_{x \to 0+} (1+2x)^{1/x}$

Solution: *This limit is almost the same as one in the notes, and in the book. The idea is to take the logarithm of the function first, so that it is in one of the standard forms ($0/0$ or ∞/∞), find the limit of that using l'Hôpital's rule, then exponentiate back to find the limit of the original function.*

$$\begin{aligned} \ln\left((1+2x)^{1/x}\right) &= \frac{1}{x}\ln(1+2x), \textit{ so} \\ \lim_{x\to 0+} \ln\left((1+2x)^{1/x}\right) &= \lim_{x\to 0+} \frac{\ln(1+2x)}{x} \; (\textit{of form } 0/0) \\ &= \lim_{x\to 0+} \frac{2/(1+2x)}{1} \\ &= 2, \end{aligned}$$

so

$$\lim_{x\to 0+} (1+2x)^{1/x} = e^2.$$

(b) $\lim\limits_{x\to\infty} \dfrac{x+\sin(3x)}{x-\sin(2x)}$

Solution: *This is of the form ∞/∞, even with the* sin() *terms. But, if you just apply l'Hôpital's rule, you run into trouble.*

$$\lim_{x\to\infty} \frac{x+\sin(3x)}{x-\sin(2x)} = \lim_{x\to\infty} \frac{1+3\cos(3x)}{1-2\sin(2x)},$$

which is a mess. That limit on the right, after applying the rule, does not exist. The denominator oscillates between -1 and 3, and the numerator separately oscillates between -2 and 4, and they are not in phase. Interesting. But this shows that, even though l'Hôpital's rule can tell you when a limit exists, it does not always work out. The theorem about the rule says that the limits are equal when the one on the right exists. Here is a case where the limit on the right does not exist, yet the one on the left is easy. All you really needed to do was *divide top and bottom by x:*

$$\begin{aligned} \lim_{x\to\infty} \frac{x+\sin(3x)}{x-\sin(2x)} &= \lim_{x\to\infty} \frac{1+\frac{\sin(3x)}{x}}{1-\frac{\sin(2x)}{x}} \\ &= 1, \end{aligned}$$

since $\sin(3x)/x$ *goes to 0 as x goes to infinity, since the numerator is bounded between* ± 1 *and the denominator is going to infinity.*

(c) $\lim\limits_{x\to 0^+} x \ln x$.

Solution: *Here you have to convert to a standard indeterminate form, since this is $0 \cdot \infty$ to begin with. You invert part of this and put it in the denominator. What part? The part that will be easier to differentiate once there, the x. If I put the* $\ln(x)$ *in the denominator, it would be a mess to differentiate the resulting* $1/\ln(x)$.

$$\begin{aligned} \lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \\ &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0^+} -\frac{x^2}{x} \\ &= 0. \end{aligned}$$

(d) $\lim\limits_{x\to 0} \dfrac{1-\cos(3x)}{x^2} =$

Solution: Apply l'Hôpital's rule, since this is of the form $\frac{0}{0}$. But once you do that, you still have a $\frac{0}{0}$ indeterminate form. It takes two applications of l'Hôpital's rule to get something which is not indeterminate.

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos(3x)}{x^2} &= \lim_{x\to 0} \frac{+3\sin(3x)}{2x} \\ &= \lim_{x\to 0} \frac{9\cos(3x)}{2} = \frac{9}{2}. \end{aligned}$$

(e) $\lim\limits_{x\to 0} \dfrac{\tan(3x)}{\sinh(2x)} =$

Solution: Apply l'Hôpital's rule, since this is of the form $\frac{0}{0}$.

$$\begin{aligned} \lim_{x\to 0} \frac{\tan(3x)}{\sinh(2x)} &= \lim_{x\to 0} \frac{3\sec^2(3x)}{2\cosh(2x)} \\ &= \frac{3}{2}. \end{aligned}$$

(f) $\lim\limits_{x\to\infty} \left(1-\dfrac{1}{x}\right)^x =$

Solution: Take the natural log, first. Then convert to a $\frac{\infty}{\infty}$form and apply l'Hôpital's rule.

$$\begin{aligned}
\ln\left(\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x\right) &= \lim_{x\to\infty}\ln\left(\left(1-\frac{1}{x}\right)^x\right)\\
&= \lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right)\\
&= \lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} \quad \text{(now in the form } \frac{\infty}{\infty}\text{)}\\
&= \lim_{x\to\infty}\frac{\left(\frac{\frac{1}{x^2}}{\left(1-\frac{1}{x}\right)}\right)}{\left(\frac{-1}{x^2}\right)}\\
&= \lim_{x\to\infty}\frac{\frac{1}{x^2}}{\left(1-\frac{1}{x}\right)}\left(\frac{x^2}{-1}\right)\\
&= \lim_{x\to\infty}\frac{-1}{\left(1-\frac{1}{x}\right)}\\
&= -1.
\end{aligned}$$

So, $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1} = \frac{1}{e}$. Of course, you can do this without our French friend's result, by recognizing that

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} &= \lim_{t\to 0}\frac{\ln(1+t)}{-t} \text{ by substituting } t=\frac{-1}{x}\\
&= -\lim_{t\to 0}\frac{\ln(1+t)-\ln(1)}{t}\\
&= -\left.(\ln(x))'\right|_{x=1} = -1.
\end{aligned}$$

However, we expect most of you would prefer l'Höpital's rule.

(g) $\lim_{x\to\infty}\dfrac{\sinh x}{3e^x+x^2-1}$.

Solution: Again l'Hôpital's rule, since this is of the form ∞/∞. Apply it 3 times to get completely rid of the x^2

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sinh x}{3e^x+x^2-1} &= \lim_{x\to\infty}\frac{\cosh x}{3e^x+2x}\\
&= \lim_{x\to\infty}\frac{\sinh x}{3e^x+2}\\
&= \lim_{x\to\infty}\frac{\cosh x}{3e^x}.
\end{aligned}$$

But at this point we need to take a fresh look at this. No matter how many times we apply l'Hôpital's rule now, we still get either $\cosh x$ or $\sinh x$ in the numerator, and $3e^x$ in the denominator. But now we can use the definition of the hyperbolic functions to say:

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sinh x}{3e^x+x^2-1} &= \lim_{x\to\infty}\frac{\cosh x}{3e^x}\\
&= \lim_{x\to\infty}\frac{\frac{1}{2}\left(e^x+e^{-x}\right)}{3e^x}\\
&= \lim_{x\to\infty}\frac{e^x+e^{-x}}{6e^x}\\
&= \lim_{x\to\infty}\frac{1}{6}+\frac{1}{6e^{2x}}.
\end{aligned}$$

Now we have to be careful once more. If this limit is $\lim_{x\to+\infty}$, then the second term goes to 0 and the limit is $\frac{1}{6}$. If the limit is $\lim_{x\to\pm\infty}$, then for the limit as $x \to -\infty$ the second term goes to infinity, and so the limit would be infinite. Only the limit as $x \to +\infty$ exists.

(h) $\lim\limits_{x\to 0^+} x(\ln x)^2$.

Solution:

$$\begin{aligned}
\lim_{x\to 0^+} x(\ln x)^2 &= \lim_{x\to 0^+}\frac{(\ln x)^2}{\left(\frac{1}{x}\right)}, \quad \text{of form } \frac{\infty}{\infty}\\
&= \lim_{x\to 0^+}\frac{2\ln(x)\frac{1}{x}}{\left(\frac{-1}{x^2}\right)}\\
&= \lim_{x\to 0^+}\frac{-2\ln(x)}{\left(\frac{1}{x}\right)} \text{ (simplify)}\\
&= \lim_{x\to 0^+}\frac{\left(\frac{-2}{x}\right)}{\left(\frac{-1}{x^2}\right)}\\
&= \lim_{x\to 0^+} 2x = 0.
\end{aligned}$$

(i) $\lim\limits_{x\to\infty}(1-\dfrac{3}{x})^x$.

Solution: As above,

$$\begin{aligned}
\ln\left(\lim_{x\to\infty}(1-\frac{3}{x})^x\right) &= \lim_{x\to\infty}\ln\left((1-\frac{3}{x})^x\right)\\
&= \lim_{x\to\infty} x\ln\left(1-\frac{3}{x}\right)\\
&= \lim_{x\to\infty}\frac{\ln\left(1-\frac{3}{x}\right)}{\left(\frac{1}{x}\right)}\\
&= \lim_{x\to\infty}\frac{\left(\frac{\left(\frac{3}{x^2}\right)}{\left(1-\frac{3}{x}\right)}\right)}{\left(\frac{-1}{x^2}\right)}\\
&= \lim_{x\to\infty}\left(\frac{\left(\frac{3}{x^2}\right)}{\left(1-\frac{3}{x}\right)}\right)(-x^2)\\
&= \lim_{x\to\infty}\left(\frac{-3}{\left(1-\frac{3}{x}\right)}\right) = -3
\end{aligned}$$

so $\lim_{x\to\infty}(1-\frac{3}{x})^x = e^{-3}$.

(40) Find an equation of the tangent line to the curve $y = e^{x^2}$ at the point where $x = 1$.

Solution: When $x = 1$, $\frac{dy}{dx} = 2xe^{x^2}\Big|_{x=1} = 2e$, and the point on the graph is $(1, e)$, so the tangent line has equation

$$\frac{y-e}{x-1} = 2e,$$

or $y = 2ex - e$.

(41) Sketch the graphs including the intervals on which the function is increasing/decreasing, the intervals on which the graph is concave upward/downward, the y-intercepts, the x-intercepts, all relative (i.e., local) extreme points, all inflection points, all absolute extreme points, and all asymptotes.

(a) $f(x) = x^4 - 4x^3 + 6x^2$

Solution: Certainly $f(0) = 0$, so $(0,0)$ is both an x- and y- intercept. But since $f(x) = x^2(x^2 - 4x + 6) = x^2((x-2)^2 + 2)$, the only place where $f(x) = 0$ is at $x = 0$. Also, $f'(x) = 4x^3 - 12x^2 + 12x = 4x(x^2 - 3x + 3)$, and again $x^2 - 3x + 3$ is irreducible (its roots are $x = \frac{3\pm\sqrt{9-12}}{2}$, which are complex, not real), so $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$, and so $(0,0)$ is the only critical point, and is a local minimum, with f increasing on $[0,\infty)$ and decreasing on $(-\infty, 0]$. $f''(x) = 12x^2 - 24x + 12 = 12(x-1)^2$, which is positive except at $x = 1$, but $x = 1$ is not a point of inflection. The curve sort of straightens out there, but does not change concavity, the graph is concave up at all points (you can quibble about whether or not the graph is concave up at $x = 1$, but really it is, since the derivative is increasing for all x). The graph looks pretty much like a parabola with vertex at $(0,0)$, opening upward, except for this (undrawable) flattening at $(1,3)$.

(b) $f(x) = x(x+1)^4$

Solution: f has no asymptotes. f has y-intercept at $y = 0$, so passes through $(0,0)$. The x-intercepts are at $x = 0$ of course, and at $x = -1$. Since $f'(x) = (x+1)^4 + 4x(x+1)^3 = (1+4x)(x+1)^3$, it has critical points at $x = -1$ and $x = -1/4$, with critical values 0 and $-\frac{3^4}{4^5}$, respectively. Since $f'(x) > 0$ for $x > -1/4$ and $x < -1$, with $f'(x) < 0$ on $(-1, -\frac{1}{4})$, f is increasing on $[-\frac{1}{4}, \infty)$ and on $(-\infty, -1]$, with f decreasing on $[-1, -\frac{1}{4}]$. $x = -1$ is a local maximum, with value 0, and $x = -1/4$ is a local minimum, with value$-\frac{3^4}{4^5}$. $f''(x) = 4(x+1)^3 + 3(1+4x)(x+1)^2 = (7+16x)(1+x)^2$, so inflection points appear to be at $(-1, 0)$ and $(-\frac{7}{16}, -\frac{7 \cdot 9^4}{16^5})$. But the concavity does not change at $x = -1$, and f is concave up on $[-\frac{7}{16}, \infty)$ and concave down on $(-\infty, -\frac{7}{16}]$. Try to sketch the graph showing all these features.

(c) $f(x) = (\ln |x|)/x$

Solution: f is not defined at $x = 0$, but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\ln |x|}{x} = -\infty$$

so $x = 0$ is a vertical asymptote, with both tails heading to $-\infty$. Since $\lim_{x\to\infty} \frac{\ln |x|}{x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$, $y = 0$ is a horizontal asymptote, approaching from above as $x \to +\infty$, and approaching from below as $x \to -\infty$. Because $x = 0$ is a vertical asymptote, f has no y-intercept. It does have x-intercepts at $x = \pm 1$, so goes through the points $(1, 0)$ and $(-1, 0)$.

$$\begin{aligned} f'(x) &= \frac{\frac{1}{x}x - \ln |x|}{x^2} \\ &= \frac{1 - \ln |x|}{x^2}, \end{aligned}$$

so $f'(x) = 0$ only when $x = \pm e$, with values $\pm\frac{1}{e}$, respectively, so the graph goes through $(e, \frac{1}{e})$ and $(-e, -\frac{1}{e})$ with horizontal tangents at those points. $f'(x) > 0$ when $x \in (-e, 0)$ or $x \in (0, e)$ and $f'(x) < 0$ on $(-\infty, -e)$ and (e, ∞).

$$\begin{aligned} f''(x) &= \frac{-\frac{1}{x}x^2 - 2x(1 - \ln |x|)}{x^4} \\ &= \frac{-3 + 2\ln |x|}{x^3}, \end{aligned}$$

so there are inflection points at $x = \pm e^{3/2}$ ($e^{3/2}$is about 4.5), with $f''(x) > 0$ for $x > e^{3/2}$, $f''(x) < 0$ for $0 < x < e^{3/2}$, $f''(x) < 0$ for $x < -e^{3/2}$, $f''(x) > 0$ for $-e^{3/2} < x < 0$ $f(\pm e^{3/2}) = \pm\frac{3}{2e^{3/2}}$. Try, again, to draw a graph illustrating those features.

(d) $f(x) = e^{2x} + 3x$

Solution: Here $f(0) = 1$ so 1 is the y-intercept. There is an x-intercept on $[-1, 0]$, since $f(-1) < 0$, but you can't exactly determine where the solution is. However, other than that $f'(x) = 2e^{2x} + 3 > 0$ so the function is increasing everywhere, and $f''(x) = 4e^{2x}$ so it is always concave up. It does have a slant-asymptote, since $\lim_{x\to-\infty} f(x) - (3x) = 0$, so $y = 3x$ is a slant-asymptote on the negative side, only. Other than that, the curve just swoops upward.

(e) $f(x) = \sin x + \cos x$.

Solution: This function has no asymptotes, but has lots of intercepts. Certainly $f(0) = 1$, so it has y-intercept 1, but it also has infinitely many x-intercepts, at all points where $\sin(x) = -\cos(x)$, which happens when $x = -\frac{\pi}{4} + n\pi$ for all integers n. Since $f'(x) = \cos(x) - \sin(x)$, it has critical points when $\cos(x) = \sin(x)$, which are at points $x = \frac{\pi}{4} + n\pi$, again, infinitely many of them. When $x = \frac{\pi}{4} + 2n\pi$ (even-integer multiples of π added on), $f(x) = \sqrt{2}$, and these are local maxima, and when $x = \frac{\pi}{4} + (2n+1)\pi$ (odd-integer multiples of π added on), $f(x) = -\sqrt{2}$, and these are local minima. Inflection points occur at all x-intercepts. But this can all be more easily seen by the following trick:

$$\begin{aligned} \sin(x) + \cos(x) &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(x) + \frac{1}{\sqrt{2}}\cos(x)\right) \\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)\sin(x) + \sin\left(\frac{\pi}{4}\right)\cos(x)\right) \\ &= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right), \end{aligned}$$

so $f(x)$ is really just a sine wave, with amplitude $\sqrt{2}$ and shifted to the left by $\pi/4$.

This trick works for any function of the form $f(x) = a\sin(x) + b\cos(x)$. Try to turn any such function into a shifted sine wave. As a hint, the amplitude is $\sqrt{a^2 + b^2}$.