MATH 21, FALL, 2008, EXAM # 1: SOLUTION

Name:__________________________________.

ID #__________________________________.

Instructions: Do all work on the test paper. ***Show all work***. You may receive no credit, even for a correct answer, if no work is shown. You may use the back if you need extra space. ***Do not*** simplify your answers, unless you are explicitly instructed to do so. This does not apply to evaluation of elementary functions at standard values or functional expressions, so that you **would** be expected to simplify $\sin(\pi/6)$ to $\frac{1}{2}$, for example. Do not write answers as decimal approximations; if $\sqrt{2}$ is the answer, leave it that way. *Except where explicitly stated otherwise, you can use the derivative rules and limit rules learned in class.*

Before giving the solutions to the specific problems, a general comment is in order. On problems like 2, 7 and to a certain extent, 4 and 6, which are not just calculations, but where you are required to explain, justify, or show, it is crucial that later statements should follow logically from earlier ones; it is not enough for every statement to turn out to be true.

(1) Find the indicated limits: Show the steps involved. *(5 points/part)*

(a) $\displaystyle\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4}$

SOLUTION: Since both numerator and denominator approach 0 as $x \to -4$, limit laws cannot be used directly; instead, the "factor and cancel" technique is called for: $x^2+5x+4 = (x+4)(x+1)$, and $x^2+3x-4 = (x+4)(x-1)$, and therefore, for all $x \neq -4$, $\dfrac{x^2+5x+4}{x^2+3x-4} = \dfrac{x+1}{x-1}$, and so

$\displaystyle\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to -4}\frac{x+1}{x-1} = \frac{-3}{-5} = \frac{3}{5}$. The next-to-last equality is by limit laws.

NOTE: This was literally Practice Problem 2.3.12 and was also quite similar to practice problems 15, 19, 20, all from Section 2.3, to problem 4 from HW 1 and the following problems from old tests: F96, Prob. 1a, F00, Prob 1a, Su01, Probs 1a, 1b, Su06, Prob 1a, Su06 Review, both 1a's, 1d.

(b) $\lim_{x\to\infty} \sqrt{x^2+2x-1} - x$

SOLUTION:
The solution proceeds along the lines of the alternate solution presented to Problem 2 on HW 2, conjugating first. Here, though, the expression that results after conjugating is much easier to deal with than in the alternate solution to Problem 2 on HW 2.
We first multiply by 1, written as $\frac{w}{w}$, where $w = \sqrt{x^2+2x-1}+x$ is the conjugate of the given expression $\sqrt{x^2+2x-1}-x$. This brings us to the limit, equivalent to the given one:

$\lim_{x\to\infty} \frac{2x-1}{\sqrt{x^2+2x-1}+x} = \frac{2x-1}{x\left(\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}+1\right)} = \frac{2-\frac{1}{x}}{\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}+1}$ (the first equality involves the same trick of factoring an x out of the radical, as in the main solution to Problem 2 in HW2), and then, by limit laws, the final limit is $\frac{2}{1+1} = 1$. Finally, rather than factoring an x out of the radical, one could also have divided numerator and denominator of $\frac{2x-1}{\sqrt{x^2+2x-1}+x}$ by x, obtaining $\frac{2-\frac{1}{x}}{\frac{\sqrt{x^2+2x-1}}{x}+1}$, and at this point, one would have to recognize that for $x > 0$:
$\frac{\sqrt{x^2+2x-1}}{x} = \sqrt{\frac{x^2+2x-1}{x^2}} = \sqrt{1+\frac{2}{x}-\frac{1}{x^2}}$.
NOTE: This was identical to 1g on the Su06 Review sheet, and is also quite similar to 1d on the Su06 test itself. The connection to Problem 2 on HW 2 has already been pointed out.

(c) $\lim_{y\to\infty} \frac{2y^2-5y-3}{5y^2+4y}$

SOLUTION: This is a more straightforward one, solved directly by dividing numerator and denominator by the highest power of y appearing in the denominator (so here, by y^2). For all $y \neq 0$:

$\frac{2y^2-5y-3}{5y^2+4y} = \frac{\frac{2y^2-5y-3}{y^2}}{\frac{5y^2+4y}{y^2}} = \frac{2-\frac{5}{y}-\frac{3}{y^2}}{5+\frac{4}{y}}$, and so:

$\lim_{y\to\infty} \frac{2y^2-5y-3}{5y^2+4y} = \lim_{y\to\infty} \frac{2-\frac{5}{y}-\frac{3}{y^2}}{5+\frac{4}{y}} = \frac{2}{5}$ (the last equality being by limit laws).
This problem was quite similar to practice problems 2.6.16, 2.6.19, to F96, 1b, F00, 1b, Su01, 1c, Su06, 1c, Su06 Review, 1c, 1f.

(2) Show that there is a solution of the equation $\cos(x) = x$ in the interval $[0, \pi/2]$. You can presume that the function $\cos(x)$ is continuous. Explicitly note which theorems you are using. *(10 points)*

SOLUTION: Various things should call to mind the Intermediate Value Theorem (IVT): the prompt to explicitly state which theorems are being used, the reminder that the cosine function is continuous, the mention of a closed interval, and what the problem is asking: show there is a root, rather than explictly finding one.

Once one realizes this, a useful next step is to state the theorem: If f is continuous on the closed interval, $[a, b]$, $f(a) \neq f(b)$, and N is any number strictly between $f(a)$ and $f(b)$, (so N is the intermediate value) then there is c in (a, b) such that $f(c) = N$ (there is a place, c, in the open interval, where the intermediate value is actually taken on as a value of the function ($f(c) = N$).

Next, note that c is a root of the equation $\cos(x) = x$ iff $\cos(c) - c = 0$, i.e., iff $f(c) = 0$, where $f(x) = \cos(x) - x$. Note that f is continuous everywhere, and so, in particular, it is continuous on $[0, \pi/2]$. Now the approach becomes clearer: we take $N = 0$, and it remains only to verify that 0 is strictly between $f(0)$ and $f(\pi/2)$. If it is, then the preceding observations mean that the hypotheses of the IVT hold for f and $[0, \pi/2]$, so the desired conclusion:

there is c in $[0, \pi/2]$ such that $f(c) = 0$,

then follows directly from the IVT. So, we calculate: $f(0) = \cos(0) - 0 = 1$, and $f(\pi/2) = \cos(\pi/2) - (\pi/2) = 0 - \pi/2 = -\pi/2$, and so, as anticipated, 0 is indeed strictly between $f(0)$ and $f(\pi/2)$.

Note that this was virtually identical to Practice Problem 2.5.49, and is easier because the complications related to $\cos(1)$ do not arise. It was also virtually identical to prob. 3 on the Su06 Review. It was also quite similar to practice problems 2.5.45, 2.5.47, and to prob. 2 on the Su06 test.

(3) Find an equation of the tangent line to the curve $y = x^4$ at the point $(2, 16)$.

SOLUTION: Using the point-slope form of the equation of a straight line, as always, the equation of the tangent line at $(2, 16)$ will have the form:

$y - 16 = m(x - 2)$, with $m = y'(2)$. **NOTE: m MUST be a NUMBER, NOT a function of x, i.e., y' MUST be evaluated at 2!!**

Here, $y'(x) = 4x^3$, so $y'(2) = 4 \cdot (2)^3 = 32$, and finally, the equation of the tangent line is $y - 16 = 32(x - 2)$.

NOTE: This was literally identical to Prob 4 on the F00 test. It was also quite similar to practice problem 3.1.34.

(4) Derivatives by the definition:

(a) Write down the definition of the derivative $f'(x)$ of a function $f(x)$ at x. *(5 points)*

SOLUTION: No formula for f was given, so none was expected, and none should be used: The definition (this should be instant reflex, by now) is:
$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

(b) Show, using the *definition* of the derivative as a limit, that $\left(5x^2+3x-2\right)' = 10x+3$. *(10 points)*

SOLUTION: Now we use the formula supplied: $f(x+h) = 5(x+h)^2+3(x+h)-2 = 5x^2+10xh+5h^2+3x+3h-2$, and therefore, $f(x+h)-f(x) = 10xh+5h^2+3h$, and so, for all $h \neq 0$:

$\dfrac{f(x+h)-f(x)}{h} = \dfrac{10xh+5h^2+3h}{h} = 10x+5h+3$. Therefore, taking the limit as $h \to \infty$:

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to\infty} 10x+5h+3 = 10x+3$, as sought.

NOTE: The wording on Part (a) was quite similar to that of 3a on F96, 5 of F00, and 4a of Su06. Part (b) was quite simlar to practice problem 2.8.23, Prob 5 on HW 2, 3b on F96, 6 on F00, 3 on Su01, 4b on Su06, 5 on Su06 Review.

(5) Find the following derivatives, using the rules we have discussed in class. *(5 points/part)*

(a) $(4x^3-x^2+1)''$

SOLUTION: First, using the power rule, the sum/difference rules, the constant rule and the constant multiple rule, we have that $\left(4x^3-x^2+1\right)' = 12x^2-2x$, and so using the power rule, the difference rule and the constant multiple, we have that $\left(4x^3-2x^2+1\right)'' = \left(\left(4x^3-2x^2+1\right)'\right)' = \left(12x^2-2x\right)' = 24x-2$.

(b) $\left(\dfrac{2x+3}{5x-1}\right)' =$

SOLUTION: By the quotient rule,
$\left(\dfrac{2x+3}{5x-1}\right)' = \dfrac{(2x+3)'(5x-1)-(2x+3)(5x-1)'}{(5x-1)^2}$, and then, since $(2x+3)' = 2$, and $(5x-1)' = 5$ (by the sum/difference, constant multiple, power and constant rules), this last expression becomes:
$\dfrac{2\cdot(5x-1)-5\cdot(2x+3)}{(5x-1)^2}$.

(c) $\left(e^x\left(x^3+4x\right)\right)' =$

SOLUTION: By the product rule,
$\left(e^x\left(x^3+4x\right)\right)' = \left(e^x\right)'\left(x^3+4x\right) + e^x\left(x^3+4x\right)'$, and since $\left(e^x\right)' = e^x$, and $\left(x^3+4x\right)' = 3x^2+4$ (the latter, using, yet again, the sum, power and constant multiple rules), this last expression becomes:
$e^x\left(x^3+4x\right) + e^x\left(3x^2+4\right)$.
NOTE: Part (a) was quite similar to practice problem 3.1.45. Problems explicitly involving second derivatives do not appear on Test 1 in previous years since they didn't occur until later in Chapter 3 in previous editios of the textbook. Parts (b), (c) were quite similar to the Practice Problems from 3.2, and to any number of the derivatives on each of the old tests.

(6) Let

$$f(x) := \begin{cases} x^2, & \text{if } x \leq 2 \\ mx+b, & \text{if } x > 2 \end{cases}.$$

Find the values of m and b for which the function f will be differentiable everywhere. *(10 points)*

SOLUTION: f will be differentiable at any $a \neq 2$ regardless of which values are chosen for m or b. This is because, letting $g(x) = x^2$, and $h(x) = mx+b$, both g and h are differentiable everywhere (since they are polynomials), and further, if $a < 2$ there is an open interval, I containing a on which f agrees with g, while if $a > 2$, there is an open inverval J containing a on which f agrees with h. So, we consider $a = 2$, and here, we'll see that the choice of m, b does matter.

The difference is that any open interval containing 2 contains both points to the left of 2 (where f agrees with g) and points to the right of 2 (where f agrees with h). Note also that $f(2) = g(2) = 4$. In order for f to be differentiable at 2 it must be continuous there. Similarly, both g and h are continuous at 2 (since they are differentiable there). Arguing as in Problem on HW , we know that in order for f to be continuous at 2 it is necessary and sufficient that
$\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$. Also,
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} g(x)$, since f agrees with g to the left of 2, and $\lim_{x\to 2^-} g(x) = g(2) = 4 = f(2)$, since g is continuous at 2 and by the definition of f.
Similarly, working on the right, we have that
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} h(x) = h(2) = 2m+b$.
Thus in order for f to be continuous at 2, it is necessary and sufficient that $2m+b = 4$. Assume, from here on, that this is true, so that f will be continuous at 2. Then, arguing as in , f will be differentiable at 2 iff the graph of $y = f(x)$ has a tangent line at the point $(2, 4)$. If it does, then the tangent line must be the same as the tangent line to the graph of $y = g(x)$ at $(2, 4)$ (since f agrees with g to the left of 2 and since g is differentiable at 2), and it must also be the same as the tangent line to the graph of $y = h(x)$ at $(2, 4)$ (since f agrees with h to the right of 2 and since h is differentiable at 2). Thus, in particular, the slope of the tangent line to the graph of $y = g(x)$ at $(2, 4)$ must be the same as the slope of the tangent line to the graph of $y = h(x)$ at $(2, 4)$, i.e. (since the slopes are given by $g'(2)$, $h'(2)$ respectively), we must have:

$2 \cdot 2 = g'(2) = h'(2) = m$, i.e, $m = 4$. Substituting this into the equation $2m + b = 4$ that expressed the continuity of f at 2, we obtain $m = 4, b = -4$.

NOTE that this was literally identical to practice problem 3.1.75 and quite similar to 3.1.67, to 2 on F96, 2 on Su01 and 4b on Su06 Review; the "continuity part" is quite similar to practice problems 2.5.35, 2.5.39, 2.5.41, 2.5.42, to Prob.4 on HW 2, and 4b on Su06 Review.

(7) Show that

$$\lim_{x \to 3} \left(x^2 + x - 4\right) = 8$$

by using an $\epsilon - \delta$ argument. *(10 points)*

SOLUTION: The limit statement will be true iff for all $\epsilon > 0$, there is $\delta > 0$ such that if $0 < |x-3| < \delta$, then $|\left(x^2 + x - 4\right) - 8| < \epsilon$, i.e, $|x^2+x-12| < \epsilon$, i.e., $|x-3| \cdot |x+4| < \epsilon$.

So, let $\epsilon > 0$ (for the remainder of the solution to this problem, this ϵ remains fixed). We will show that there is $\delta > 0$, depending only on ϵ (and implicitly on 3) such that if $0 < |x-3| < \delta$ then $|x-3| \cdot |x+4| < \epsilon$. In the first phase of our work, we are led to the choice of a value for δ, after which we verify that this choice of δ "meets the ϵ challenge." The key step in the first phase is to make a preliminary choice of δ_1, with the idea that (for reasons which will become clear later) we will end up requiring that $\delta \leq \delta_1$. The point of this is to find a constant, c such that $|x+4| < c$ for all x in $(3 - \delta_1,\ 3 + \delta_1)$. Here, (essentially because $x+4$ is defined for all x), we have total freedom in choosing δ_1, and so, for convenience, we choose $\delta_1 = 1$. This means we are considering the behavior of $|x+4|$ on the interval $(2,4)$, and more particularly, we wish to find a constant c such that $|x+4| < c$ for all x in $(2,4)$.

To this end, note that if $2 < x < 4$ then (adding 4 to the entire inequality), $6 < x+4 < 8$. Thus, $x + 4 \geq 0$ for all x in $(2,4)$, and so, for all such x, $|x+4| = x + 4 < 8$, i.e., we take $c = 8$. But then, for all such x, $|x-3| \cdot |x+4| < 8|x-3|$, and therefore, in order to be sure that $|x-3| \cdot |x+4| < \epsilon$, it will suffice to know that ($2 < x < 4$, i.e., $|x-3| < 1$ and) $8|x-3| < \epsilon$ (since then $|x-3| \cdot |x+4| < 8|x-3| < \epsilon$. But the requirement that $8|x-3| < \epsilon$ can be rewritten as $|x-3| < \epsilon/8$. Thus we take $\delta_2 = \epsilon/8$, and we are led to our choice of δ:

$$\delta = \min\left(\delta_1,\ \delta_2\right) = \min(1,\ \epsilon/8).$$

To verify that this choice of δ works, suppose that $0 < |x-3| < \delta$. Then, $|x-3| < 1$ and $|x-3| < \epsilon/8$. Since $|x-3| < 1$, by the above argument, $|x+4| < 8$, But then, $|x-3| \cdot |x+4| < 8|x-3| < 8 \cdot (\epsilon/8) = \epsilon$.

NOTE: This was quite similar to practice problems 29, 30, 31, 36 (all from Section 2.4), to Prob 3 from HW 2, and to 7 of Su06, and to 8 of Su06 Review.

(8) A ball is tossed up in the air so that its height above the ground t seconds after being tossed is $s(t) = -16t^2 + 32t + 5$ feet. *(15 points)*

(a) How fast was the ball moving at the instant when it was tossed?

SOLUTION: The instant the ball was tossed is the time $t = 0$ (0 seconds after it was tossed), and how fast it was moving at that instant is the velocity of the ball at time $t = 0$. Since $s(t)$ the height at time t, $s'(t)$ gives the velocity at time t, and therefore, we are asked to find $s'(0)$. By the usual derivative rules (sum/difference, power, constant multiple and constant rules), $s'(t) = -32t+32$, and therefore, $s'(0) = (-32 \cdot 0) + 32 = 32$ (with units of ft/sec).

(b) How high was the ball above the ground one second after it was tossed?

SOLUTION: By reasonsing similar to that given in part (a), here we see that we are asked to find $s(1) = (-16 \cdot 1^2) + 32 \cdot 1 + 5 = 16 + 5 = 21$ (with units of ft).

(c) What was its instantaneous velocity at $t = 1$?

SOLUTION: By reasonsing similar to that given in part (a), here we see that we are asked to find $s'(1)$. Since we saw, in part (a) that $s'(t) = -32t + 32$, $s'(1) = (-32 \cdot 1) + 32 = 0$ (with units of ft/sec).
NOTE that Parts (a), (b) are literally identical to Parts (a), (b) of 6 of Su06, and part (c) here is an easier version of part (c) of that problem. This was also quite similar to practice problems 2.7.13, 2.7.16 (only easier, since you were not required to find the derivative by the definition, as the directions made quite clear), to 5 on F96, and t0 7 on Su06 Review.