Math 21 Fall 2005, Lehigh University, Exam III review solutions.

1. (i) Call the lengths on the base x and the height h. The volume is $V = x^2h$ and thus $h = V/x^2$. The material for the box consists of the top and bottom, each with area x^2 and the 4 sides, each with area xh. So the total material is $M = 2x^2 + 4xh$. Substituting $h = V/x^2$ we get $M = 2x^2 + 4x(V/x^2) = 2x^2 + 4V/x$. Then $M' = 4x - 4V/x^2$. This is 0 when $4x = 4V/x^2 \Rightarrow x^3 = V$. So $x = \sqrt[3]{V}$ is a possible minimum. To observe that this is an absolute minimum (and hence the answer) note that $M'' = 4 + 8V/x^3$ is positive for all $x > 0$ so M is concave up for all $x > 0$ and thus the the local minimum is an absolute minimum for $x > 0$.

   (ii) Call the width x so the base has dimensions x by $3x$. Call the height h. The volume is $8 = x(3x)h = 3x^2h$ (cubic feet) so $h = \dfrac{8}{3x^2}$. The top and bottom have area $3x^2$ (square feet) and hence cost $(3)(3x^2) = 9x^2$ (dollars). Multiply by two to get the cost of both top and bottom equal to $18x^2$. Two sides have area xh and two have area $3xh$ for a total area of $2(xh) + 2(3xh) = 8xh$ (square feet) and hence total cost for the sides is $8xh$ (dollars). So the cost is $C = 18x^2 + 8xh$. Substituting $h = 8/(3x^2)$ we get $C = 18x^2 + 8xh = 18x^2 + 8x\dfrac{8}{3x^2} = 18x^2 + \dfrac{64}{3x}$. Then $C' = 36x - \dfrac{64}{3x^2}$ and $C' = 0$ when $36x = \dfrac{64}{3x^2} \Rightarrow x^3 = \dfrac{16}{27}$ and hence $x = \dfrac{2\sqrt[3]{2}}{3}$. To observe that this is an absolute minimum (and hence the answer) note that $C'' = 36 + 192/x^3$ is positive for all $x > 0$ so C is concave up for all $x > 0$ and thus the the local minimum is an absolute minimum for $x > 0$.

2. (i) The marginal cost is $C'(x) = 20 + 2x$. The average cost is $c(x) = C(x)/x = 100/x + 20 + x$. To minimize average cost note that $c'(x) = -100/x^2 + 1$ which is 0 when $x^2 = 100 \Rightarrow x = \pm 10$. Since we must have $x > 0$ we take $x = 10$. To check that this is an absolute minimum for $x > 0$ note that $c''(x) = 200/x^3$ is always positive for $x > 0$. Thus the average cost is concave up for $x > 0$ and the local minimum at $x = 10$ is an absolute minimum. Thus the production level to minimize average cost is $x = 10$.

   (ii) Note the change to $p(x) = 38 - 2x$. We have revenue $R(x) = xp(x) = 38x - 2x^2$ and profit $P(x) = R(x) - C(x) = (38x - 2x^2) - (100 + 20x + x^2) = -3x^2 + 18x - 100$. Then $P'(x) = -6x + 18$ which is 0 when $x = 3$. Since $P''(x) = -6$ we see that $P(x)$ is always concave down and the local maximum at $x = 3$ is also an absolute maximum.

   (iii) The linear demand function contains the points $(48, 12)$ and $(64, 10)$ and thus has slope $(10-12)/(64-48) = -1/8$. Then using the point slope equation for a line we have $\dfrac{-1}{8} = \dfrac{p(x) - 10}{x - 64} \Rightarrow p(x) = 10 + \dfrac{-1}{8}(x - 64) = \dfrac{-x}{8} + 18$. Then revenue is $R(x) = xp(x) = -x^2/8 + 18x$ and with $C(x) = 6x$ (since each of x widgets produced costs \$6) we have profit $P(x) = R(x) - C(x) = -x^2/8 + 18x - 6x = -x^2/8 + 12x$. Then $P'(x) = -x/4 + 12$ which is 0 when $x = 48$. To check that this a maximum for $x > 0$ note that $P''(x) = -1/4$ so $P(x)$ is always concave down and the local maximum at $x = 48$ is an absolute maximum. The selling price for production level $x = 48$ that maximizes profit is $p(48) = -48/8 + 18 = 12$ (dollars).

3. Use Newton's method formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ with $x_1 = 2$, $f(x) = x^2 - 8x$ and $f'(x) = 2x - 8$. With $f(2) = 2^2 - 8(2) = -12$ and $f'(2) = 2(2) - 8 = -4$ we get $x_2 = 2 - f(2)/f'(2) = 2 - (-12)/(-4) = -1$. Then $f(-1) = (-1)^2 - 8(-1) = 9$ and $f'(-1) = 2(-1) - 8 = -10$ and $x_3 = x_2 - f(x_2)/f'(x_2) = -1 - f(-1)/f'(-1) = -1 - 9/(-10) = -1/10$.

4. (i) With $f(x) = x^2$ we have $f'(x) = 2x$. Thus the slope of the tangent at $(3, 9)$ is $f'(3) = 2 \cdot 3 = 6$. Then an equation for the tangent line at $(3, 9)$ is $6 = \frac{y-9}{x-6}$. The x-intercept is where $y = 0$. Setting $y = 0$ in the tangent line equation and solving for x we get $6 = \frac{0-9}{x-3} \Rightarrow x = 9$.

   (ii) With $f(x) = x^2$ we have $f'(x) = 2x$. Thus the slope of the tangent at (x_1, x_1^2) is $f'(x_1) = 2x_1$. Then an equation for the tangent line at (x_1, x_1^2) is $2x_1 = \frac{y-x_1^2}{x-x_1}$. The x-intercept is where $y = 0$. Setting $y = 0$ in the tangent line equation and solving for x we get $2x_1 = \frac{0-x_1^2}{x-x_1} \Rightarrow x = \frac{x_1}{2}$.

   (iii) The slope of the tangent at $(x_1, f(x_1))$ is $f'(x_1)$. An equation for the tangent line at $(x_1, f(x_1))$ is $f'(x_1) = \frac{y-f(x_1)}{x-x_1}$. The x-intercept is where $y = 0$. Setting $y = 0$ in the tangent line equation and solving for x we get $f'(x_1) = \frac{0-f(x_1)}{x-x_1} \Rightarrow x = x_1 - \frac{f(x_1)}{f'(x_1)}$.
   This is just the Newton's method formula.

5. (i) The general antiderivative of $f(x) = 6\sqrt[5]{x^7} - \dfrac{8}{x^3} + x^{-4/7} = 6x^{7/5} - 8x^{-3} + x^{-4/7}$ is $F(x) = 6\dfrac{x^{7/5+1}}{7/5+1} - 8\dfrac{x^{-3+1}}{-3+1} + \dfrac{x^{-4/7+1}}{-4/7+1} + C = \dfrac{5}{2}x^{12/5} + \dfrac{4}{x^2} + \dfrac{7}{3}x^{3/7} + C$.

   (ii) The general antiderivative of $f(x) = \cos(x) - e^x + \dfrac{6}{x^2+1} - 9 + \sec^2(x)$ is $F(x) = \sin(x) - e^x + 6\tan^{-1}(x) - 9x + \tan(x) + C$.

6. If $f''(x) = 8$ then $f'(x) = 8x + C$ and $4 = f'(2) = 8(2) + C \Rightarrow 4 = 16 + C \Rightarrow C = -12$. So $f'(x) = 8x - 12$ and $f(x) = 4x^2 - 12x + D$ and $10 = f(2) = 4(2^2) - 12(2) + D \Rightarrow 10 = -8 + D \Rightarrow D = 18$. Thus $f(x) = 4x^2 - 12x + 18$.

7. If $s''(t) = a(t) = a$ then $s'(t) = v(t) = at + C$ and $v_0 = s'(0) = v(0) = a(0) + C \Rightarrow C = v_0$. So $s'(t) = v(t) = at + v_0$ and $s(t) = \frac{1}{2}at^2 + v_0 t + D$ and

$s_0 = s(0) = \frac{1}{2}0^2 + v_0(0) + D \Rightarrow s_0 = D$. Thus $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$.

8. Use $\sum_{i=1}^{n} 1 = n$, $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{2n^3+3n^2+n}{6}$. Then

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\left(2+\frac{2i}{n}\right)^2 &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\left(4+\frac{8i}{n}+\frac{4i^2}{n^2}\right)\\
&= \lim_{n\to\infty}\frac{24}{n}\sum_{i=1}^{n}1+\frac{48}{n^2}\sum_{i=1}^{n}i+\frac{24}{n^3}\sum_{i=1}^{n}i^2\\
&= \lim_{n\to\infty}\frac{24}{n}(n)+\frac{48}{n^2}\left(\frac{n^2+n}{2}\right)+\frac{24}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)\\
&= \lim_{n\to\infty}\left(24+(24+\frac{24}{n})+(8+\frac{12}{n}+\frac{6}{n^2})\right)\\
&= 24+24+0+8+0+0\\
&= 56
\end{aligned}
$$

9. The interval of width $6 = 8-2$ is divided into 3 pieces of width $6/3 = 2$ so the right endpoints of the rectangles are $2+2 = 4, 2+(2)(2) = 6, 2+(2)(3) = 8$. The heights of the rectangles are $f(4) = 4^3 = 64, f(6) = 6^3 = 216, f(8) = 8^3 = 512$. So the area is approximated by $2(64+216+512) = 1584$.
10. Divide the interval of width $2 = 4-2$ into n pieces of width $2/n$. The i^{th} rectangle has right endpoint $2+2i/n$ and height $f(2+2i/n) = 3(2+2i/n)^2$ so we have the area equal to $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}3\left(2+\frac{2i}{n}\right)^2$.
11. This is the area under $3x^2$ for $2 \le x \le 4$ as in the previous problem.
12. Here are three possible solutions. For each, when we write down the expression for the Reimann sum in the definition of the definite integral using right endpoints and equal area rectangles the result is $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(\sin\left(7+\frac{5i}{n}\right)+\left(13+\frac{5i}{n}\right)^3\right)$.

$\int_7^{12}\frac{3}{5}(\sin x + (6+x)^3)\,dx$
$\int_{13}^{18}\frac{3}{5}(\sin(-6+x)+x^3)\,dx$
$\int_{21/5}^{36/5}(\sin(\frac{5}{3}x)+(6+\frac{5}{3}x)^3)\,dx$

13. We must evaluate the sum arising in problems 8, 10, 11 which by the answer to 8 is 56.
14. The region under $2x$ for $-5 \le x \le 5$ is two triangles each with base length 5 and height $2\cdot 5 = 10$. One is below the axis and one above so in the integral the areas cancel. Thus $\int_{-5}^{5} 2x\,dx = 0$. The region under $\sqrt{25-x^2}$ for $-5 \le x \le 5$ is the upper half of a the circle of radius 5 centered at the origin. This semicircle has area $\frac{1}{2}\pi 5^2 = \frac{25\pi}{2}$. Thus $\int_{-5}^{5}\sqrt{25-x^2}\,dx = \frac{25\pi}{2}$.

So $\int_{-5}^{5} 2x - \sqrt{25-x^2}\, dx = \int_{-5}^{5} 2x\, dx - \int_{-5}^{5} \sqrt{25-x^2}\, dx = 0 - \frac{25\pi}{2}$.

15. For $-1 \le x \le 1$ we have $0 \le x^2 \le 1$ so $1 \le 1+x^2 \le 2$ thus $f(x) = \sqrt{1+x^2}$ has maximum value $\sqrt{2}$ and minimum $\sqrt{1} = 1$ on the interval $-1 \le x \le 1$. (We could also determine the maximum and minimum values for $f(x)$ on this interval using methods of calculus as we have done previously.) Then since $1 \le \sqrt{1+x^2} \le \sqrt{2}$ on $-1 \le x \le 1$ we have $2 = (1-(-1))(1) \le \displaystyle\int_{-1}^{1} \sqrt{1+x^2}dx \le (1-(-1))(\sqrt{2}) = 2\sqrt{2}$.

16. It would be more clear to state this as $g(x) = \int_2^x t(t^2+7)^9\, dt$.
(i) Directly from Part I we get $g'(x) = x(x^2+7)^9$
(ii) Use substitution with $u = t^2+7$ and $du = 2tdt$. So we get $\displaystyle\int_{t=2}^{t=x} u^9\, du/2 = \left[\frac{u^{10}}{20}\right]_{t=2}^{t=x} = \left[\frac{(t^2+7)^{10}}{20}\right]_{t=2}^{t=x} = \frac{(x^2+7)^{10}}{20} - \frac{(2^2+7)^{10}}{20} = \frac{(x^2+7)^{10}}{20} - \frac{11^{10}}{20}$. The derivative of $\frac{(x^2+7)^{10}}{20} - \frac{11^{10}}{20}$ is $10\frac{(x^2+7)^9}{20}(2x+0) - 0 = x(x^2+7)^9$.

17. From part I of the fundamental theorem and the chain rule $f'(x) = ((\sin x^2)^3 - 7(\sin x^2))(\sin x^2)' = ((\sin x^2)^3 - 7(\sin x^2))(\cos x^2)(2x)$.

18. (i) From the net change theorem it is the increase in the child's weight in pounds between the ages of 5 and 10.
(ii) From the net change theorem it is the increase in revenue when production is increased from 1000 to 5000 units.

19. (i) $\displaystyle\int_{27}^{125} \frac{1}{\sqrt[3]{x}}\, dx = \int_{27}^{125} x^{-1/3}\, dx = \left[\frac{3x^{2/3}}{2}\right]_{27}^{125} = \frac{3(125)^{2/3}}{2} - \frac{3(27)^{2/3}}{2} = \frac{75}{2} - \frac{27}{2} = 24$.
(ii) $\displaystyle\int_{-8}^{27} \frac{1}{\sqrt[3]{x}}\, dx$ does not exist because $\frac{1}{\sqrt[3]{x}}$ is not continuous on $[-8, 27]$ since it is not defined at $x = 0$.
(iii) $\displaystyle\int_{-\pi}^{\pi} \sin\theta\, d\theta = [-\cos\theta]_{-\pi}^{\pi} = -\cos(\pi) - (-\cos(-\pi)) = -(-1) + (-1) = 0$.
(iv) $\displaystyle\int_{-\pi}^{\pi} \cos\theta\, d\theta = [\sin(\theta)]_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi) = 0 - 0 = 0$.
(v) $\displaystyle\int_{-2}^{7} 6\, dx = [6x]_{-2}^{7} = (6)(7) - (6)(-2) = 42 + 12 = 54$.
(vi) $\displaystyle\int_0^r \pi(r^2 - x^2)\, dx = \pi\left[r^2x - \frac{x^3}{3}\right]_0^r = \pi\left(\left((r^2)(r) - \frac{r^3}{3}\right) - \left((r^2)(0) - \frac{0^3}{3}\right)\right) = \pi\left(r^3 - \frac{r^3}{3}\right) = \frac{2\pi r^3}{3}$.

(vii) $\int_0^h \left(\frac{-b}{h}y + b\right)\, dy = \left[\frac{-b}{h}\frac{y^2}{2} + by\right]_0^h = \left(\frac{-b}{h}\frac{h^2}{2} + bh\right) - \left(\frac{-b}{h}\frac{0^2}{2} + (b)(0)\right) = \frac{-bh^2}{2} + bh = \frac{bh}{2}$.

(viii) $\int_1^2 \frac{dx}{(3x-5)^2}$ does not exist because $\frac{dx}{(3x-5)^2}$ is not continuous on $[1, 2]$ since it is not defined at $x = 5/3$.

A similar problem not on the review sheet for which the integral exists is $\int_2^3 \frac{dx}{(3x-5)^2}$. Let $u = 3x - 5$ so $du = 3\, dx$. Then $\int_2^3 \frac{dx}{(3x-5)^2} = \int_{x=2}^{x=3} \frac{du/3}{u^2} = \int_{x=2}^{x=3} \frac{u^{-2}}{3}\, du = \left[-\frac{u^{-1}}{3}\right]_{x=2}^{x=3} = \left[\frac{-1}{3u}\right]_{x=2}^{x=3} = \left[\frac{-1}{3(3x-5)}\right]_2^3 = \frac{-1}{3(9-5)} - \frac{-1}{3(6-5)} = \frac{1}{4}$.

(ix) $\int_1^e \frac{\ln x}{x}\, dx$ is example 9 section 5.5.

(x) Assume that $a > 0$. Let $u = x^2 + a^2$ so $du = 2x\, dx$ and $x\, dx = du/2$. When $x = a$ we have $u = a^2 + a^2 = 2a^2$ and when $x = 0$ we have $u = 0^2 + a^2 = a^2$. Then $\int_0^a x\sqrt{x^2 + a^2}\, dx = \int_{a^2}^{2a^2} \sqrt{u}\, du/2 = \int_{a^2}^{2a^2} \frac{u^{1/2}}{2}\, du = \left[\frac{u^{3/2}}{3}\right]_{a^2}^{2a^2} = \frac{(2a^2)^{3/2}}{3} - \frac{(a^2)^{3/2}}{3} = \frac{2\sqrt{2}a^3}{3} - \frac{a^3}{3} = \frac{a^3}{3}(2\sqrt{2} - 1)$.

20. (i) $\int 6\, dx = 6x + C$.

(ii) $\int \tan\theta\, d\theta$ is example 6 of section 5.5.

(iii) For the first term let $u = 5x^3 + 9$ then $du = 15x^2\, dx$, for the second and third terms let $u = e^x$ then $du = e^x\, dx$. Separating terms and writing $e^{2x} = (e^x)^2$ in the last term we get $= \int 2x^2 e^{5x^3+9}\, dx + \int \frac{e^x}{1+e^x}\, dx + \int \frac{e^x}{1+(e^x)^2}\, dx = \int e^u \frac{2du}{15} + \int \frac{1}{1+u}\, du + \int \frac{1}{1+u^2}\, du = \frac{2e^u}{15} + \ln|u| + \tan^{-1}(u) + C = \frac{2e^{5x^3+9}}{15} + \ln|e^x| + \tan^{-1}(e^x) + C$.

(iv) Let $u = t^3 + 9t$ then $du = 3t^2 + 9$ so $du/3 = t^2 + 3$. Then $\int (t^2 + 3)\cos(t^3 + 9t)\, dt = \int \cos(u)\, \frac{du}{3} = \frac{-\sin(u)}{3} + C = \frac{-\sin(t^3 + 9t)}{3} + C$.

(v) Write $x^4 = (x^2)^2$ and let $u = x^2$ so $du = 2x\, dx$. Then $= \int \frac{x}{1+(x^2)^2}\, dx =$

$\displaystyle\int \frac{du/2}{1+u^2} = \frac{\tan^{-1}(u)}{2} + C = \frac{\tan^{-1}(x^2)}{2} + C.$

(vi) Let $u = 2ax^2+16$ so $du = 4ax\,dx$ and $3du/2 = 6ax\,dx$. Then $\displaystyle\int \frac{6ax}{\sqrt[5]{2ax^2+16}}\,dx = \int \frac{3du/2}{\sqrt[5]{u}} = \int \frac{3u^{-1/5}}{2}\,du = \frac{3}{2}\frac{5u^{4/5}}{4} + C = \frac{15}{8}(2ax^2+16)^{4/5} + C = \frac{15}{8}\sqrt[5]{(2ax^2+16)^4} + C$.

(vii) Let $u = \cos(t^2)$ so (using the chain rule) $du = -\sin(t^2)(2t)\,dt$. Also writing $\cos^5(t^2) = (\cos(t^2))^5$ we get $= \displaystyle\int 2t\sin(t^2)(\cos(t^2))^5\,dt = \int u^5\,(-du) = -\frac{u^6}{6} + C = -\frac{\cos^6(t^2)}{6} + C.$

(viii) Let $u = x^2+7$ so $du = 2x\,dx$ and also $x^2 = u-7$. Then $\displaystyle\int x^3(x^2+7)^4\,dx = \int x^2(x^2+7)^4(x\,dx) = \int (u-7)u^4\frac{du}{2} = \int \left(\frac{u^5}{2} - \frac{7u^4}{2}\right)du = \frac{u^6}{12} + \frac{7u^5}{10} + C = \frac{(x^2+7)^6}{6} + \frac{7(x^2+7)^5}{10} + C.$

(ix) $\displaystyle\int \left(2\sqrt[9]{x^7} - \frac{5}{x^{12}}\right)dx = \int \left(2x^{7/9} - 5x^{-12}\right)dx = \frac{2x^{16/9}}{16/9} - \frac{5x^{-11}}{-11} + C = \frac{9x^{16/9}}{8} + \frac{5}{11x^{11}} + C.$

(x) Assume that $a \neq -1$ and $c \neq 0$. Let $u = b + cx^{a+1}$ so $du = c(a+1)x^a\,dx$ and $\displaystyle\frac{du}{c(a+1)} = x^a\,dx$. Then $\displaystyle\int x^a\sqrt{b+cx^{a+1}}\,dx = \int \sqrt{u}\frac{du}{c(a+1)} = \int u^{1/2}\frac{du}{c(a+1)} = \frac{2u^{3/2}}{3c(a+1)} + C = \frac{2(b+cx^{a+1})^{3/2}}{3c(a+1)} + C.$

21. (i) The line between $(-2,2)$ and $(6,6)$ has slope $\displaystyle\frac{6-2}{6-(-2)} = \frac{4}{8} = \frac{1}{2}$ and hence equation $\displaystyle\frac{1}{2} = \frac{y-2}{x-(-2)} \Rightarrow y = \frac{1}{2}x + 3$. Similarly, we get the line between $(-2,2)$ and $(3,-3)$ to be $y = -x$ and the line between $(3,-3)$ and $(6,6)$ to be $y = 3x - 12$.

Using vertical rectangles the area is divided into two parts depending on the bounding lines and is given by the integral $\displaystyle\int_{-2}^{3}\left((\frac{1}{2}x+3) - (-x)\right)dx + \int_{3}^{6}\left((\frac{1}{2}x+3) - (3x-12)\right)$

For horizontal rectangles we first rewrite the equation of each line for x in terms of y getting $x = 2y - 6$, $x = -y$ and $x = \frac{1}{3}y + 4$. The area is given by the integral $\displaystyle\int_{-3}^{2}\left((\frac{1}{3}y+4) - (-y)\right)dy + \int_{2}^{6}\left((\frac{1}{3}y+4) - (2y-6)\right)dy.$

(ii) The line between $(0,0)$ and (c,h) has equation $x = \frac{c}{h}y$ and the line between (c,h) and $(b,0)$ has equation $x = \frac{c-b}{h}y + b$. (These can be determined as in part (i).) Using horizontal rectangles the area is given by the integral $\int_0^h \left((\frac{c-b}{h}y + b) - (\frac{c}{h}y) \right) dy = \int_0^h \left(\frac{-b}{h}y + b \right) dy$. The value of this integral was determined in 4(vii) to be $\frac{bh}{2}$.

22. (i) The curves intersect when $9x^2 = x^4 \Rightarrow 9 = x^2 \Rightarrow 3 = x$ since $x \geq 0$. So they intersect at the point $(3, 81)$. Using vertical rectangles the area is $\int_0^3 (9x^2 - x^4)\, dx = \left[3x^3 - \frac{x^5}{5} \right]_0^3 = (3(3^3) - \frac{3^5}{5}) - (3(0^3) - \frac{0^5}{5}) = (81 - \frac{243}{5}) - (0 - 0) = \frac{162}{5}$.
For horizontal rectangles rewrite the bounding curves as $x = y^{1/4}$ and $x = y^{1/2}/3$. Then the area is $\int_0^{81} (y^{1/4} - y^{1/2}/3)\, dy = \left[\frac{4y^{5/4}}{5} - \frac{2y^{3/2}}{9} \right]_0^{81} = \frac{4(81)^{5/4}}{5} - \frac{2(81)^{3/2}}{6} = \frac{4(3^5)}{5} - \frac{2(9^3)}{9} = \frac{4(243)}{5} - 162 = \frac{162}{5}$.
(ii) The curves intersect when $x^2 - 4 = 2x + 4 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0 \Rightarrow x = 4, -2$. Using vertical rectangles the area is $\int_{-2}^4 \left((2x+4) - (x^2 - 4) \right) dx = \int_{-2}^4 (-x^2 + 2x + 8)\, dx = \left[\frac{-x^3}{3} + x^2 + 8x \right]_{-2}^4 = (\frac{-(4^3)}{3} + 4^2 + 8(4)) - (\frac{-(-2)^3}{3} + (-2)^2 + 8(-2)) = (\frac{-64}{3} + 16 + 32) - (\frac{8}{3} + 4 - 16) = 36$.

23. (i)

(ii) Using vertical rectangles $\int_0^{27} (3 - x^{1/3})\, dx$.

Using horizontal rectangles $\int_0^3 y^3\, dy$.

(iii) Rotate about $y = 0$ (x-axis): washers - $\int_0^{27} (\pi 3^2 - \pi(x^{1/3})^2)\, dx$: shells - $\int_0^3 2\pi y(y^3)\, dy$.

Rotate about $y = 3$: washers - $\int_0^{27} \pi(3 - x^{1/3})^2\, dx$: shells - $\int_0^3 2\pi(3-y)(y^3)\, dy$.

Rotate about $y = 5$: washers - $\int_0^{27} \left(\pi(5 - x^{1/3})^2 - \pi(5-3)^2 \right) dx$: shells - $\int_0^3 2\pi(5-y)(y^3)\, dy$.

Rotate about $x = 0$ (y-axis): washers - $\int_0^3 \pi(y^3)^2\, dy$: shells - $\int_0^{27} 2\pi x(3 - x^{1/3})\, dx$.

Rotate about $x = -3$: washers - $\int_0^3 \left(\pi(y^3)^2 - \pi 3^2\right) dy$: shells - $\int_0^{27} 2\pi(x + 3)(3 - x^{1/3})\, dx$.

Rotate about $x = 27$: washers - $\int_0^3 \left(\pi 27^2 - \pi(27 - y^3)^2\right) dy$: shells - $\int_0^{27} 2\pi(27 - x)(3 - x^{1/3})\, dx$.

24. (i) To get the cone rotate the triangle with vertices $(0,0)$, $(0,h)$ and $(r,0)$ (bounded by the x-axis, the y-axis and the line $y = \frac{-h}{r}x + h$) about the y-axis. Using shells the volume is $\int_0^r 2\pi x(\frac{-h}{r}x + h)\, dx = 2\pi \int_0^r (\frac{-h}{r}x^2 + hx)\, dx = 2\pi \left[\frac{-h}{3r}x^3 + \frac{h}{2}x^2\right]_0^r = 2\pi(\frac{-hr^3}{3r} + \frac{hr^2}{2} - 0) = 2\pi h(\frac{r^2}{3} + \frac{r^2}{2}) = 2\pi h(\frac{r^2}{6} = \frac{\pi r^2 h}{3}$.
Using disks we would evaluate the integral $\int_0^h \pi(\frac{-r}{h}y + r)^2\, dy$.

(ii) Rotating as described in the problem and using disks the volume is $\int_{-r}^r \pi(\sqrt{r^2 - x^2})^2\, dx = \pi \int_{-r}^r (r^2 - x^2)\, dx = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^r = \pi\left(\left(r^2(r) - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right) = \pi\left(\frac{2r^3}{3} - \frac{-2r^3}{3}\right) = \frac{4\pi r^3}{3}$.

25.