Math 21 Fall 2005, Lehigh University, Exam III review

These are a sampling of problems for review for the final exam. This review covers only material not on the first two reviews. The final exam is comprehensive, covering all material from the semester. This sample of problems may not be complete, be sure to prepare from all material from class and homework assignments.

Remember that use of calculators will not be allowed during the exam.

1. Solve the following optimization problems. Be sure to justify your answer.
   (i) A box with square base is made to contain volume $V\ m^3$. Find the dimensions that minimize the amount of material needed to make the box.
   (ii) A box has its length equal to three times its width. Material for the sides costs \$1 per square foot and material for the top and bottom costs \$3 per square foot. If the volume is to be 8 cubic feet, find the cost of materials for the cheapest such box.

2. (i) For cost function $C(x) = 100 + 20x + x^2$ find the marginal cost, average cost and production level to minimize average cost.
   (ii) For cost function $C(x) = 100 + 20x + x^2$ and demand function $p(x) = 38 - 2x$ find the production level that will maximize profit.
   (iii) A company selling widgets at \$12 each averages 48 sales per day. During a sale they sell the widgets at \$10 and average 64 sales per day. Assuming a linear demand function (and no other effects from there being a sale) what price should the widgets be sold at to maximize profit if each widget costs \$6 to produce?
3. Use Newton's method with $x_1 = 2$ to find x_3, the third approximation to the root of $f(x) = x^2 - 8x$.
4. (i) For $f(x) = x^2$ determine the x-intercept of the tangent line at $(3, 9)$.
   (ii) For $f(x) = x^2$ determine, in terms of x_1, the x-intercept of the tangent line at (x_1, x_1^2).
   (iii) For a given function $f(x)$, determine, in terms of x_1, $f(x_1)$ and $f'(x_1)$ the x-intercept of the tangent line at $(x_1, f(x_1))$. Be sure to show your work in deriving this.
5. Find the most general antiderivative for the following:
   (i) $f(x) = 6\sqrt[5]{x^7} - \dfrac{8}{x^3} + x^{-4/7}$
   (ii) $\cos(x) - e^x + \dfrac{6}{x^2+1} - 9 + \sec^2(x)$
6. If $f''(x) = 8$ with $f'(2) = 4$ and $f(2) = 10$ what is $f(x)$?
7. Show that for motion in a straight line with constant acceleration a, initial velocity v_0 and initial displacement s_0, the displacement after time t is $s(t) = \dfrac{1}{2}at^2 + v_0 t + s_0$.
8. Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n}\left(2 + \frac{2i}{n}\right)^2$. Use $\displaystyle\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = \frac{2n^3+3n^2+n}{6}$.
9. Estimate the area under $f(x) = x^3$ from $x = 2$ to $x = 8$ using 3 approximating rectangles and right endpoints.
10. Find an expression for the area under $f(x) = 3x^2$ for $2 \le x \le 4$ as a limit of a sum.

11. Determine a region whose area is equal to $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n}\left(2+\frac{2i}{n}\right)^2$.

12. Express $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\sin\left(7+\frac{5i}{n}\right)+\left(13+\frac{5i}{n}\right)^3\right)$ as a definite integral in two different ways.

13. Use the definition of the definite integral to evaluate $\int_2^4 3x^2 dx$

14. Evaluate $\int_{-5}^{5} 2x - \sqrt{25-x^2}\, dx$ by interpreting it in terms of areas.

15. Use bounds on $f(x) = \sqrt{1+x^2}$ to show that $2 \le \int_{-1}^{1} \sqrt{1+x^2}\, dx \le 2\sqrt{2}$.

16. For $g(x) = \int_2^x x(x^2+7)^9\, dx$ find $g'(x)$ in two ways: (i) Use part I of the Fundamental Theorem of Calculus (ii) Use part II of the Fundamental Theorem of Calculus and then differentiate.

17. Find $f'(x)$ if $f(x) = \int_{-2}^{\sin x^2} (t^3 - 7t)\, dt$.

18. (i) If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\, dt$ represent?
(ii) If $R'(t)$ is marginal revenue, the derivative of revenue $R(x)$, with x the number of units sold, what does $\int_{1000}^{5000} R'(x)\, dx$ represent?

19. For each of the following definite integrals evaluate it or explain why it does not exist.

(i) $\int_{27}^{125} \frac{1}{\sqrt[3]{x}}\, dx$	(ii) $\int_{-8}^{27} \frac{1}{\sqrt[3]{x}}\, dx$
(iii) $\int_{-\pi}^{\pi} \sin\theta\ d\theta$	(iv) $\int_{-\pi}^{\pi} \cos\theta\ d\theta$
(v) $\int_{-2}^{7} 6\, dx$	(vi) $\int_0^r \pi(r^2 - x^2)\, dx$
(vii) $\int_0^h \left(\frac{-b}{h}y + b\right)\, dy$	(viii) $\int_1^2 \frac{dx}{(3x-5)^2}$
(ix) $\int_1^e \frac{\ln x}{x}\, dx$	(x) $\int_0^a x\sqrt{x^2+a^2}\, dx$

20. For each of the following indefinite integrals determine its most general form.

(i) $\displaystyle\int 6\,dx$ (ii) $\displaystyle\int \tan\theta\,d\theta$

(iii) $\displaystyle\int \left(2x^2e^{5x^3+9} + \frac{e^x}{1+e^x} + \frac{e^x}{1+e^{2x}}\right) dx$ (iv) $\displaystyle\int (t^2+3)\cos(t^3+9t)\,dt$

(v) $\displaystyle\int \frac{x}{1+x^4}\,dx$ (vi) $\displaystyle\int \frac{6ax}{\sqrt[5]{2ax^2+16}}$

(vii) $\displaystyle\int 2t\sin(t^2)\cos^5(t^2)\,dt$ (viii) $\displaystyle\int x^3(x^2+7)^4\,dx$

(ix) $\displaystyle\int \left(2\sqrt[9]{x^7} - \frac{5}{x^{12}}\right) dx$ (x) $\displaystyle\int x^a\sqrt{b+cx^{a+1}}\,dx$

21. (i) Set up, but do not evaluate, integrals that give the area of the triangle with vertices $(-2,2),(6,6),(3,-3)$ using both vertical and horizontal rectangles.
(ii) For $0<c<b$ use calculus to determine the area of the triangle with vertices $(0,0),(b,0),(c,h)$.

22. (i) Sketch the area in the first quadrant ($x \geq 0$ and $y \geq 0$) between the curves $y=9x^2$ and $y=x^4$ and determine its area using both vertical and horizontal rectangles.
(ii) Sketch the area between the $y=x^2-4$ and $y=2x=4$ and determine its area.

23. (i) Sketch the area bounded by $y=\sqrt[3]{x}$, $y=3$ and $x=0$ (the y-axis).
(ii) Set up, but do not evaluate integrals which give the area of the region in (i) using both vertical and horizontal rectangles.
(iii) Set up, but do not evaluate, two integrals (using the method of cylindrical shells for one and the method of disks/washers for the other) that give the volume of the solid of revolution obtained by rotating the region in part (i) about the following: the x-axis, the y-axis, $y=3$, $y=5$, $x=-3$, $x=27$.

24. (i) Use calculus to show that the volume of a right circular cone with height h and base radius r is $\dfrac{\pi r^2 h}{3}$.
(ii) Use calculus to show that the volume of a sphere with radius r is $\dfrac{4\pi r^3}{3}$. Note that the equation of a semicircle (with $y \geq 0$) with radius r centered at the origin is $y=\sqrt{r^2-x^2}$ and that rotating the region under this curve and above the x-axis about the x-axis gives a sphere of radius r.

25. If a cup of coffee has temperature 95°C in a room where the temperature is 20°C, then, according to Newton's Law of Cooling, the temperature of the coffee after t minutes is $T(t)=20+75e^{-t/50}$. What is the average temperature of the coffee during the first half hour?