Classical Control

Topics covered:

Modeling. ODEs. Linearization.

Laplace transform. Transfer functions.

Block diagrams.

Time response specifications.

Effects of zeros and poles.

Stability.

Feedback: Disturbance rejection, Sensitivity, Steady-state tracking.

PID controllers and Ziegler-Nichols tuning procedure.

Classical Control

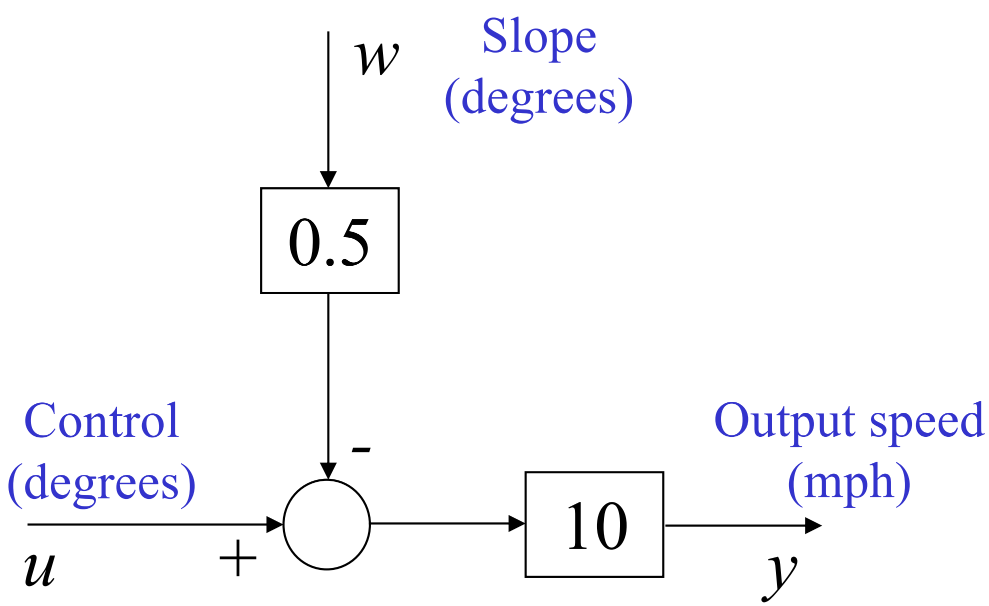
Text: *Feedback Control of Dynamic Systems,*
8th Edition, G.F. Franklin, J.D. Powel and A. Emami-Naeini
Prentice Hall 2002.

What is control?

For any analysis we need a mathematical MODEL of the system

Model → Relation between gas pedal and speed:
10 mph change in speed per each degree rotation of gas pedal
Disturbance → Slope of road:
5 mph change in speed per each degree change of slope

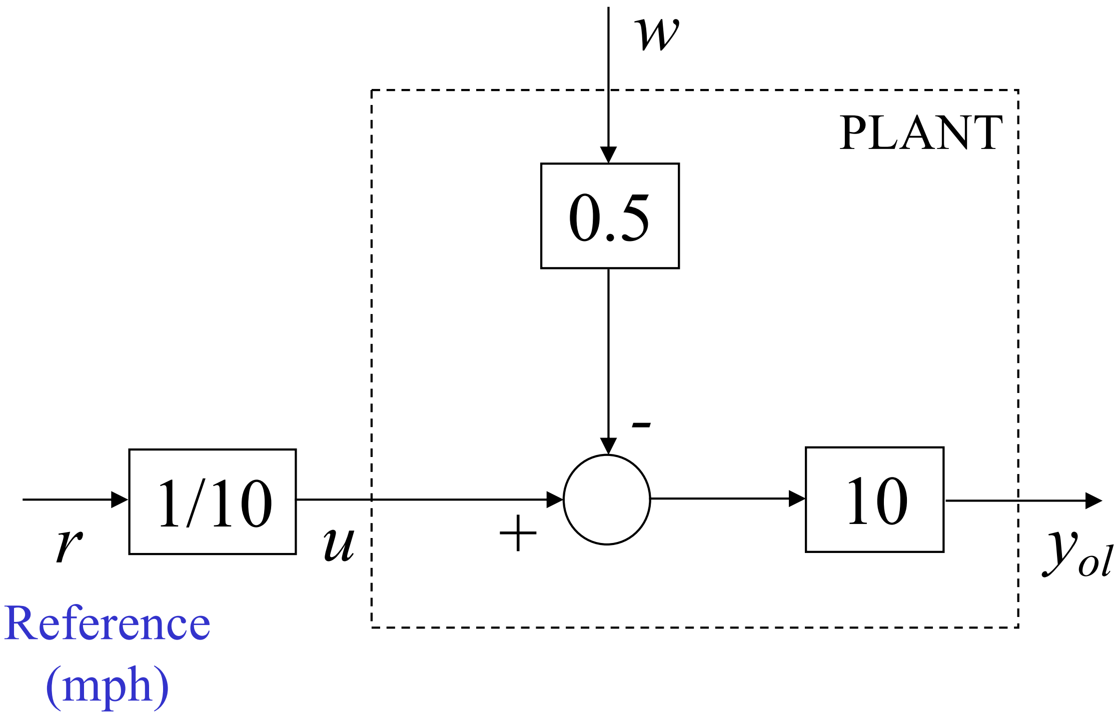
Block diagram for the cruise control "plant":



$$y = 10(u - 0.5w)$$

What is control?

Open-loop cruise control:



$$u = \frac{r}{10}$$

$$y_{ol} = 10(u - 0.5w)$$

$$= 10(\frac{r}{10} - 0.5w)$$

$$= r - 5w$$

$$e_{ol} = r - y_{ol} = 5w$$

$$e_{ol}[\%] = \frac{r - y_{ol}}{r} = 500\frac{w}{r}$$

$r = 65, w = 0 \Rightarrow e_{ol} = 0$

$r = 65, w = 1 \Rightarrow e_{ol} = 5\text{mph}, e_{ol} = 7.69\%$

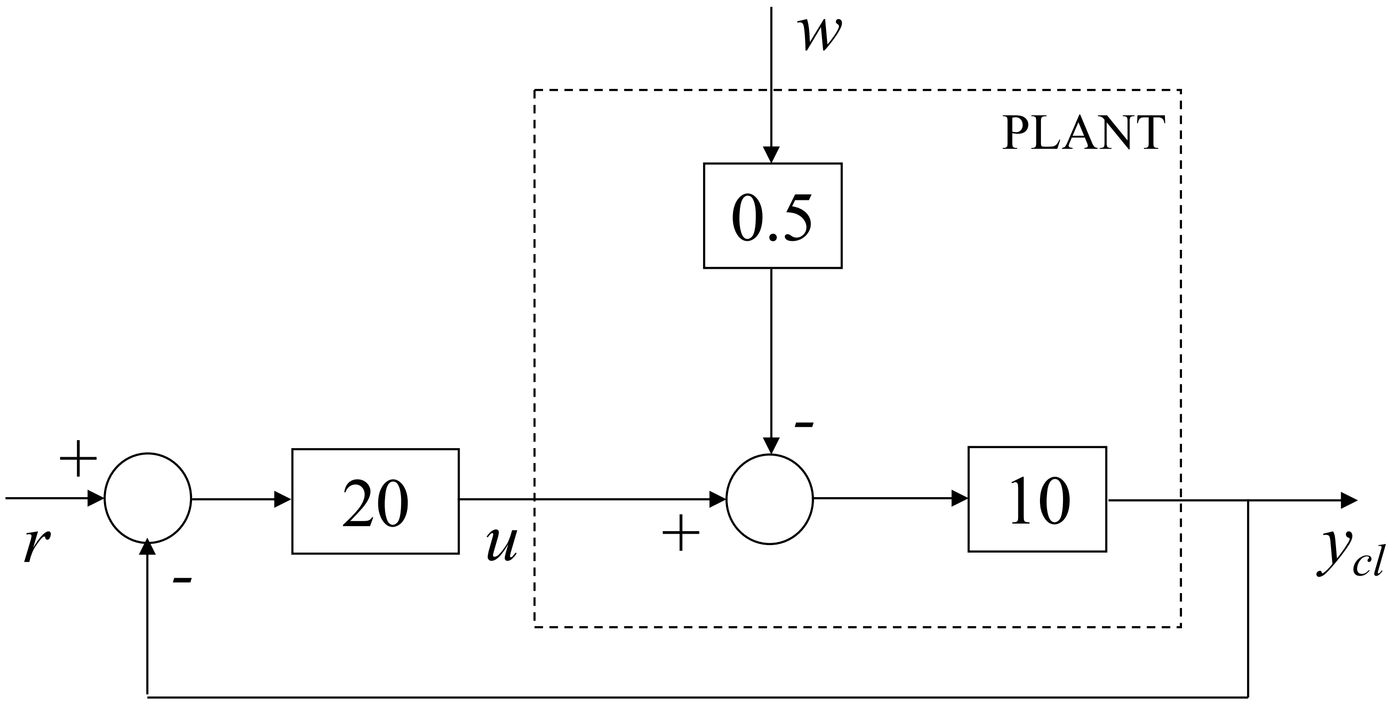
OK when:

1- Plant is known exactly

2- There is no disturbance

What is control?

Closed-loop cruise control:



$$u = 20\left(r - y_{cl}\right)$$

$$y_{cl} = 10(u - 0.5w)$$

$$= \frac{200}{201}r - \frac{5}{201}w$$

$$e_{cl} = r - y_{cl} = \frac{1}{201}r + \frac{5}{201}w$$

$$e_{cl}[\%] = \frac{r - y_{cl}}{r} = \frac{1}{201} + \frac{5}{201}\frac{w}{r}$$

$$r = 65, w = 0 \Rightarrow e_{cl} = \frac{1}{201}\% \approx 0.5\%$$

$$r = 65, w = 1 \Rightarrow e_{cl} = \frac{1}{201} + \frac{5}{201}\frac{1}{65}\% \approx 0.69\%$$

What is control?

Feedback control can help:

- reference following (tracking)
- disturbance rejection
- changing dynamic behavior

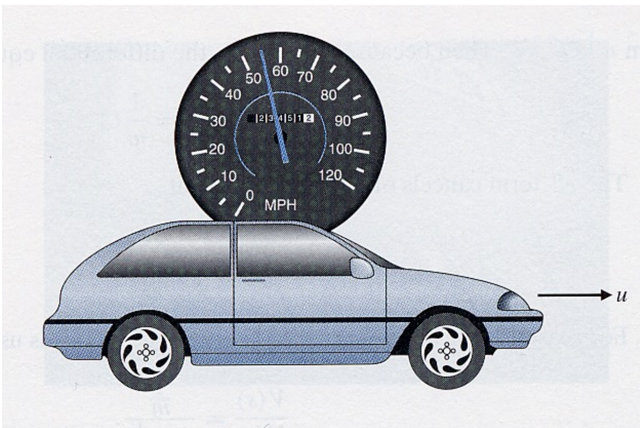
LARGE gain is essential but there is a STABILITY limit

"The issue of how to get the gain as large as possible to reduce the errors due to disturbances and uncertainties without making the system become unstable is what much of feedback control design is all about"

First step in this design process: DYNAMIC MODEL

Dynamic Models

MECHANICAL SYSTEMS: $F = ma$ Newton's law



$$m\ddot{x} = u - b\dot{x}$$

$v = \dot{x}$ velocity

$a = \dot{v} = \ddot{x}$ acceleration

$$\dot{v} = -\frac{b}{m}v + \frac{u}{m} \xrightarrow[v = V_o e^{st},\, u = U_o e^{st}]{} \frac{V_o}{U_o} = \frac{1/m}{s + b/m}$$

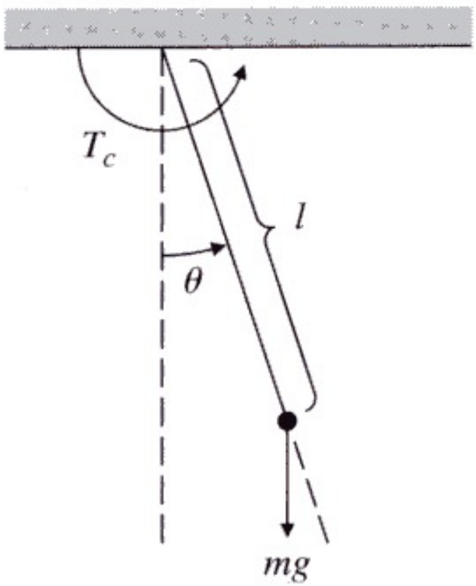
Transfer Function

$$\frac{d}{dt} \rightarrow s$$

How do we compute the transfer function?

Dynamic Models

MECHANICAL SYSTEMS: $F \Box I\alpha$ Newton's law



$$ml^2\ddot{\theta} \Box -lmg\sin\theta \Box T_c$$

$\omega \Box \dot{\theta}$ angular velocity

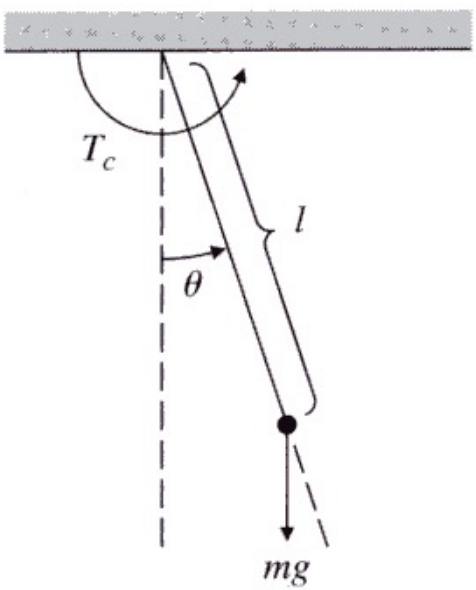
$\alpha \Box \dot{\omega} \Box \ddot{\theta}$ angular acceleration

$I \Box ml^2$ moment of inertia

$$\ddot{\theta} \Box \frac{g}{l}\sin\theta \Box \frac{T_c}{ml^2} \xrightarrow[\sin\theta\approx\theta]{} \ddot{\theta} \Box \frac{g}{l}\theta \Box \frac{T_c}{ml^2}$$ Linearization

How do we linearize the nonlinear dynamics?

Dynamic Models



$$\ddot{\theta} = \frac{g}{l}\theta + \frac{T_c}{ml^2}$$

Reduce to first order equations:

$$x_1 = \theta \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{\theta} \qquad \dot{x}_2 = -\frac{g}{l}x_1 + \frac{T_c}{ml^2}$$

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u \equiv \frac{T_c}{ml^2} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

State Variable Representation

General case:

$$\dot{x} = Fx + Gu$$

$$y = Hx + Ju$$

Dynamic Models

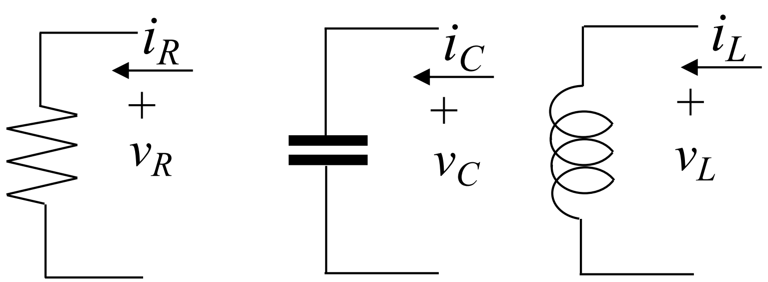
ELECTRICAL SYSTEMS:

Kirchoff’s Current Law (KCL):

The algebraic sum of currents entering a node is zero at every instant

Kirchoff’s Voltage Law (KVL)

The algebraic sum of voltages around a loop is zero at every instant



Resistors:

$$v_R(t) \square R i_R(t) \Leftrightarrow i_R(t) \square G v_R(t)$$

Capacitors:

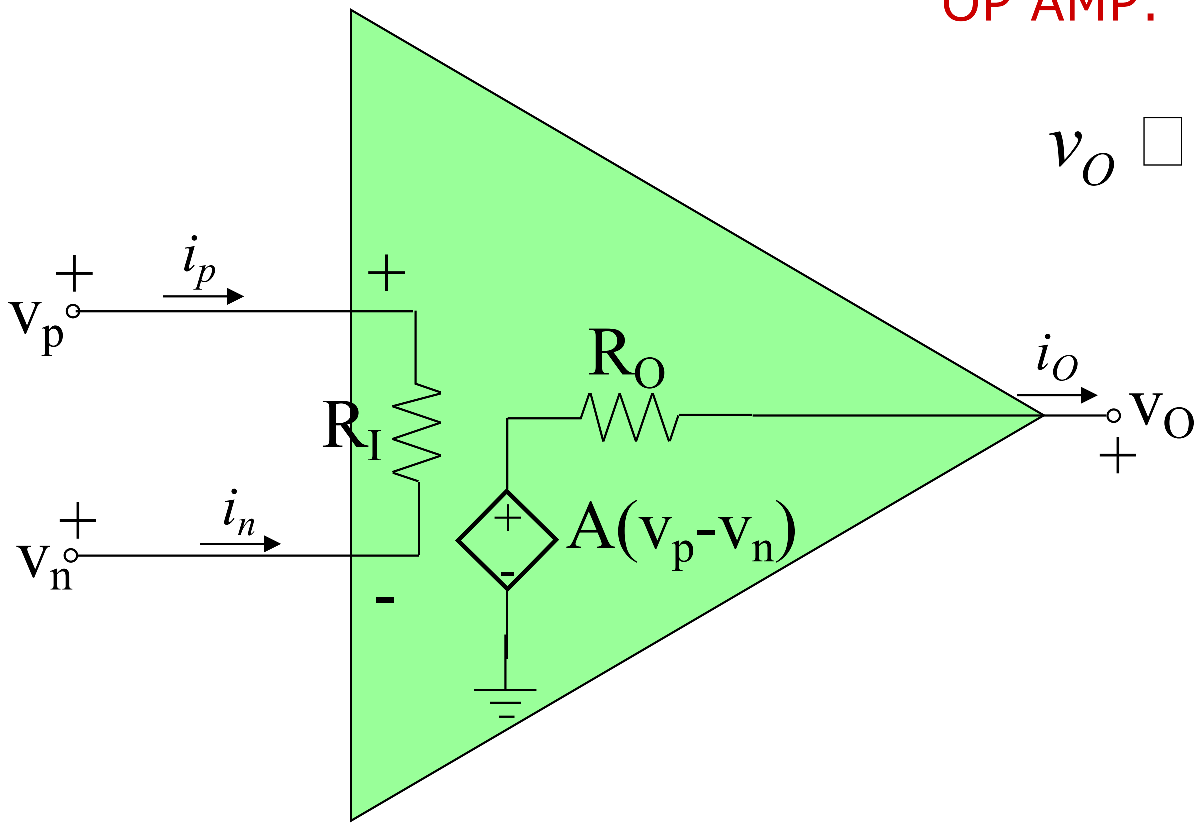
$$i_C(t) \square C \frac{dv_C(t)}{dt} \Leftrightarrow v_C(t) \square \frac{1}{C}\int_0^t i_C(\tau)d\tau \square v_C(0)$$

Inductors:

$$v_L(t) \square L \frac{di_L(t)}{dt} \Leftrightarrow i_L(t) \square \frac{1}{L}\int_0^t v_L(\tau)d\tau \square i_L(0)$$

Dynamic Models

ELECTRICAL SYSTEMS:



OP AMP:

$$v_O \square A(v_p - v_n), A \to \infty$$

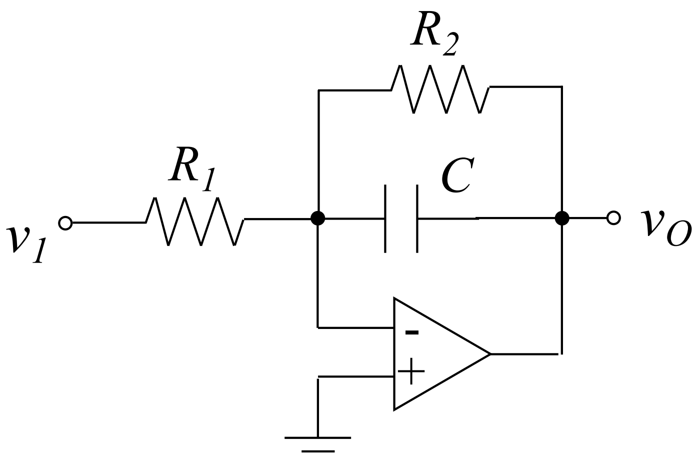
$$v_p \square v_n$$

$$i_p \square i_n \square 0$$

To work in the linear mode we need FEEDBACK!!!

Dynamic Models

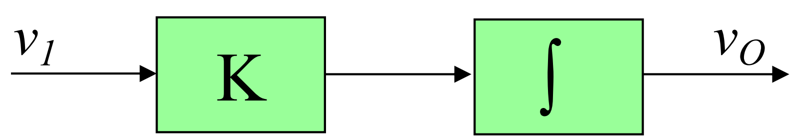
ELECTRICAL SYSTEMS:



KCL:

$$\frac{dv_O}{dt} = -\frac{1}{R_2C}v_O - \frac{1}{R_1C}v_I$$

$$R_2 \to \infty\ (\mathrm{OC}) \Rightarrow v_O(t) = v_O(0) - \frac{1}{R_1C}\int_0^t v_I(\tau)d\tau$$

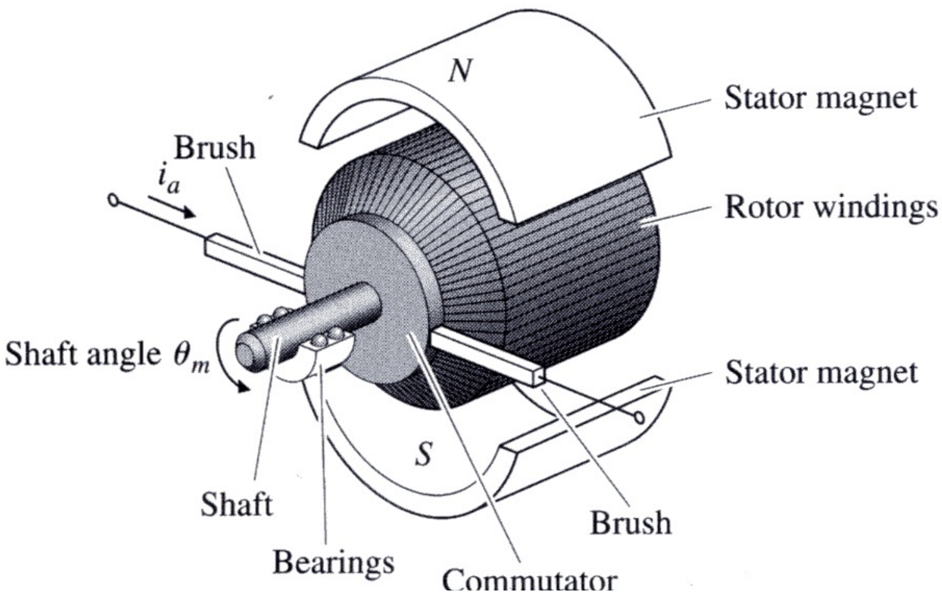


$$K = -\frac{1}{RC}$$

Inverting integrator

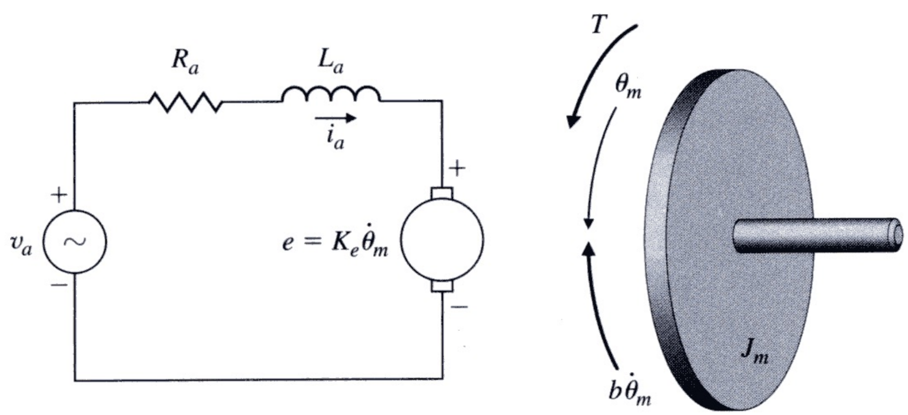
Dynamic Models

ELECTRO-MECHANICAL SYSTEMS: DC Motor



torque → $T = K_t i_a$ ← armature current

emf → $e = K_e \dot{\theta}_m$ ← shaft velocity



(a) (b)

$$J_m \ddot{\theta}_m = -b\dot{\theta}_m + T$$

$$-v_a + R_a i_a + L\frac{di_a}{dt} + e = 0$$

Obtain the State Variable Representation

Dynamic Models

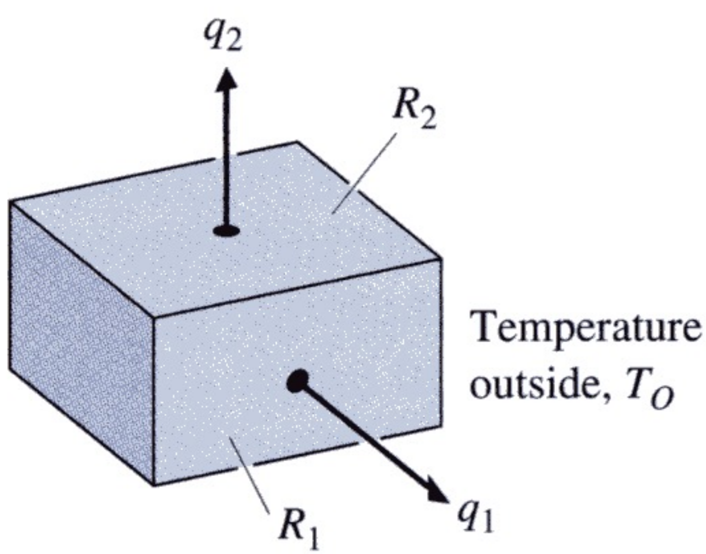
HEAT-FLOW:

Heat Flow — Temperature Difference

$$q = \frac{1}{R}\left(T_1 - T_2\right)$$

$$\dot{T} = \frac{1}{C} q$$

Thermal capacitance — Thermal resistance



$$\dot{T}_I = \frac{1}{C_I}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(T_o - T_I\right)$$

Dynamic Models

FLUID-FLOW:

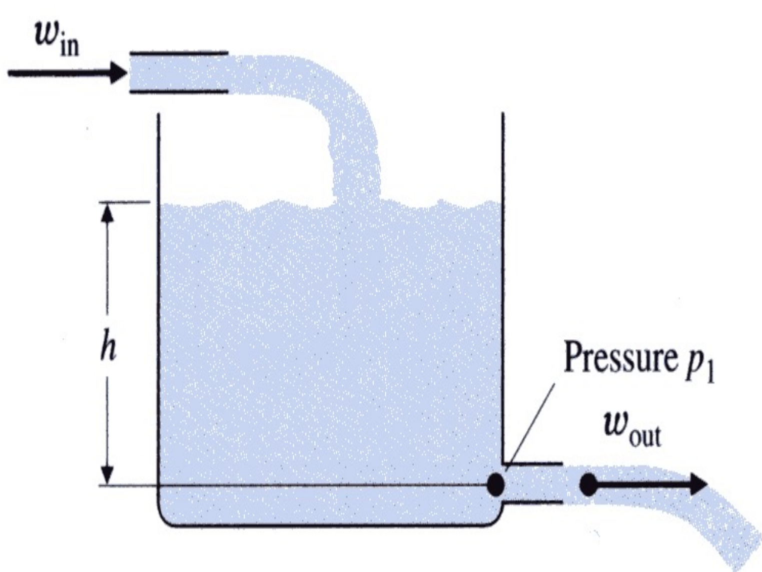
Mass rate

Mass Conservation law

$$\dot{m} = w_{in} - w_{out}$$

Inlet mass flow

Outlet mass flow



$$\dot{m} = \rho A \dot{h} \Rightarrow \dot{h} = \frac{1}{\rho A}\left(w_{in} - w_{out}\right)$$

A: area of the tank

ρ: density of fluid

h: height of water

Linearization

Dynamic System: $\dot{x} = f(x, u)$

$$0 = f(x_o, u_o)$$ Equilibrium

Denote $\delta x = x - x_o, \delta u = u - u_o$

$$\delta \dot{x} = f(x_o + \delta x, u_o + \delta u)$$

Taylor Expansion

$$\delta \dot{x} \approx f(x_o, u_o) + \left.\frac{\partial f}{\partial x}\right|_{x_o, u_o} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x_o, u_o} \delta u$$

$$F \equiv \left.\frac{\partial f}{\partial x}\right|_{x_o, u_o}, G \equiv \left.\frac{\partial f}{\partial u}\right|_{x_o, u_o} \Rightarrow \quad \delta \dot{x} \approx F \delta x + G \delta u$$

Linearization

$$\delta\dot{x} \approx F\delta x + G\delta u$$

$$F \equiv \left.\frac{\partial f}{\partial x}\right|_{x_o,u_o} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x_o,u_o}, \quad G \equiv \left.\frac{\partial f}{\partial u}\right|_{x_o,u_o} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{x_o,u_o}$$

Example: Pendulum with friction

$$\ddot{\theta} + \frac{k}{m}\dot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}_{x_o} x$$

Laplace Transform

- Function *f(t)* of time
  - Piecewise continuous and exponential order $|f(t)| \le Ke^{bt}$

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}^{-1}\left[F(s)\right] = f(t) = \frac{1}{2\pi j}\int_{\alpha-j\infty}^{\alpha+j\infty} F(s)e^{st}ds$$

  - *0-* limit is used to capture transients and discontinuities at *t=0*
  - *s* is a complex variable ($\sigma + j\omega$)
    - There is a need to worry about regions of convergence of the integral
  - Units of *s* are sec^{-1}=Hz
    - A frequency
  - If *f(t)* is volts (amps) then *F(s)* is volt-seconds (amp-seconds)

Laplace transform examples

- Step function – unit Heavyside Function
  - After Oliver Heavyside (1850-1925)

$$u(t) = \begin{cases} 0, \text{ for } t < 0 \\ 1, \text{ for } t \geq 0 \end{cases}$$

$$F(s) = \int_{0-}^{\infty} u(t) e^{-st} dt = \int_{0-}^{\infty} e^{-st} dt = -\frac{e^{-st}}{s}\bigg|_0^{\infty} = -\frac{e^{-(\sigma + j\omega)t}}{\sigma + j\omega}\bigg|_0^{\infty} = \frac{1}{s} \text{ if } \sigma > 0$$

- Exponential function
  - After Oliver Exponential (1176 BC- 1066 BC)

$$F(s) = \int_{0}^{\infty} e^{-\alpha t} e^{-st} dt = \int_{0}^{\infty} e^{-(s+\alpha)t} dt = -\frac{e^{-(s+\alpha)t}}{s+\alpha}\bigg|_0^{\infty} = \frac{1}{s+\alpha} \text{ if } \sigma > \alpha$$

- Delta (impulse) function *δ(t)*

$$F(s) = \int_{0-}^{\infty} \delta(t) e^{-st} dt = 1 \text{ for all } s$$

Example: ME207_Laplace.m

Laplace Transform Pair Tables

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Laplace Transform Properties

Linearity: (absolutely critical property)

$$\mathcal{L}\{Af_1(t) + Bf_2(t)\} = A\mathcal{L}\{f_1(t)\} + B\mathcal{L}\{f_2(t)\} = AF_1(s) + BF_2(s)$$

Integration property:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Differentiation property:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0-)$$

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2F(s) - sf(0-) - f'(0-)$$

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^mF(s) - s^{m-1}f(0-) - s^{m-2}f'(0-) - \cdots - f^{(m)}(0-)$$

Laplace Transform Properties

Translation properties:

s-domain translation: $\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$

t-domain translation: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ for $a \ge 0$

Initial Value Property:

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

Final Value Property:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

If all poles of $F(s)$ are in the LHP

Laplace Transform Properties

Time Scaling: $$\mathcal{L}\{f(at)\} = \frac{1}{|a|}F(\frac{s}{a})$$

Multiplication by time: $$\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$$

Convolution: $$\mathcal{L}\{\int_0^t f(\tau)g(t-\tau)d\tau\} = F(s)G(s)$$

Time product: $$\mathcal{L}\{f(t)g(t)\} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)G(s-\lambda)d\lambda$$

Laplace Transform

Exercise: Find the Laplace transform of the following waveform

$$f(t) = \left[2 + 2\sin(2t) - 2\cos(2t)\right]u(t) \qquad F(s) = \frac{4(s+2)}{s(s^2+4)}$$

Exercise: Find the Laplace transform of the following waveform

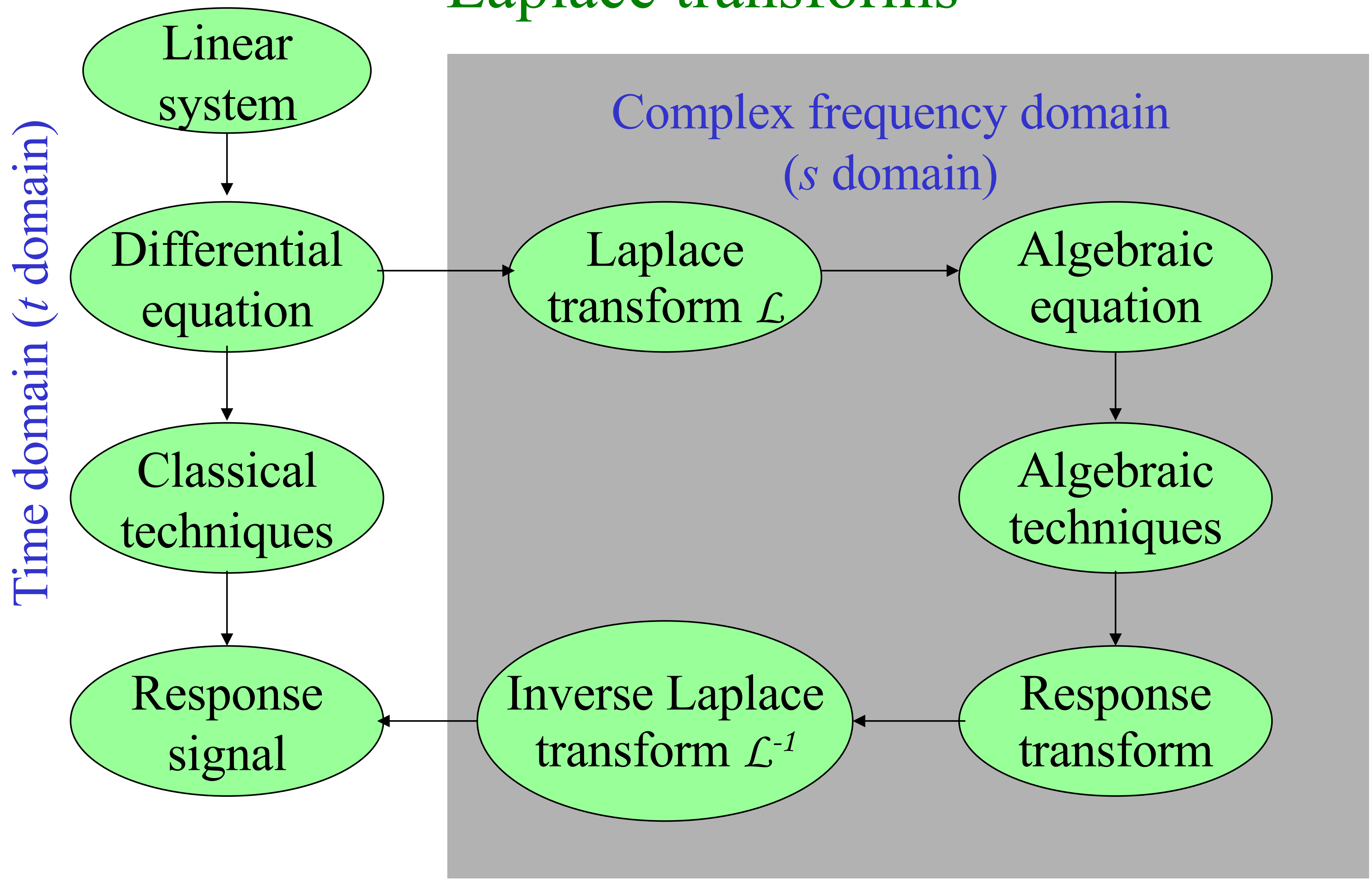
$$f(t) = e^{-4t}u(t) + 5\int_0^t \sin(4x)\,dx \qquad F(s) = \frac{s^3 + 36s + 80}{s(s+4)(s^2+16)}$$

$$f(t) = 5e^{-40t}u(t) + \frac{d\left(5te^{-40t}\right)}{dt}u(t) \qquad F(s) = \frac{10s + 200}{(s+40)^2}$$

Exercise: Find the Laplace transform of the following waveform

$$f(t) = Au(t) - 2Au(t-T) + Au(t-2T) \qquad F(s) = \frac{A\left(1 - e^{-Ts}\right)^2}{s}$$

Laplace transforms



- The diagram commutes
  - Same answer whichever way you go

Solving LTI ODE's via Laplace Transform

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

Initial Conditions: $\quad y^{(n-1)}(0),\ldots,y(0),u^{(m-1)}(0),\ldots,u(0)$

Recall $\quad \mathcal{L}\left\{\frac{d^k f(t)}{dt^k}\right\} = s^k F(s) - \sum_{j=0}^{k-1} f^{(k-1-j)}(0)s^j$

$$s^n Y(s) - \sum_{j=0}^{n-1} y^{(n-1-j)}(0)s^j + \sum_{i=0}^{n-1} a_i\left[s^i Y(s) - \sum_{j=0}^{i-1} y^{(i-1-j)}(0)s^j\right] = \sum_{i=0}^{m} b_i\left[s^i U(s) - \sum_{j=0}^{i-1} u^{(i-1-j)}(0)s^j\right]$$

$$Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}U(s) + \frac{\sum_{i=0}^{n-1} a_i \sum_{j=0}^{i-1} y^{(i-1-j)}(0)s^j - \sum_{i=0}^{m} b_i \sum_{j=0}^{i-1} u^{(i-1-j)}(0)s^j}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

For a given rational $U(s)$ we get $Y(s)=Q(s)/P(s)$

Laplace Transform

Exercise: Find the Laplace transform $V(s)$

$$\frac{dv(t)}{dt} + 6v(t) = 4u(t) \qquad V(s) = \frac{4}{s(s+6)} - \frac{3}{s+6}$$

$$v(0-) = -3$$

Exercise: Find the Laplace transform $V(s)$

$$\frac{d^2v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 3v(t) = 5e^{-2t} \qquad V(s) = \frac{5}{(s+1)(s+2)(s+3)} - \frac{2}{s+1}$$

$$v(0-) = -2, v'(0-) = 2$$

What about $v(t)$?

Transfer Functions

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

Assume all Initial Conditions Zero:

$$\left[s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right]Y(s) = \left[b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right]U(s)$$

Output → $Y(s)$, Input → $U(s)$

$$Y(s) = \frac{b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}U(s) = \frac{B(s)}{A(s)}U(s)$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Rational Functions

- We shall mostly be dealing with LTs which are rational functions – ratios of polynomials in s

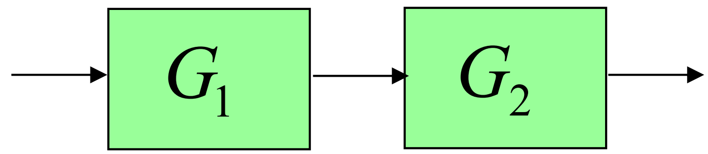
$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

- p_i are the poles and z_i are the zeros of the function
- K is the scale factor or (sometimes) gain
- A proper rational function has $n \geq m$
- A strictly proper rational function has $n > m$
- An improper rational function has $n < m$

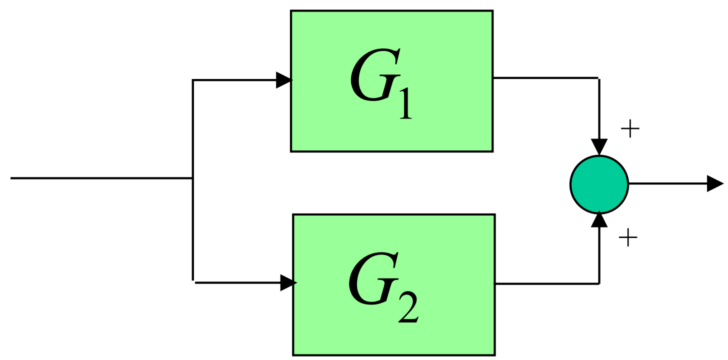
Block Diagrams

Series:



$$G = G_1 G_2$$

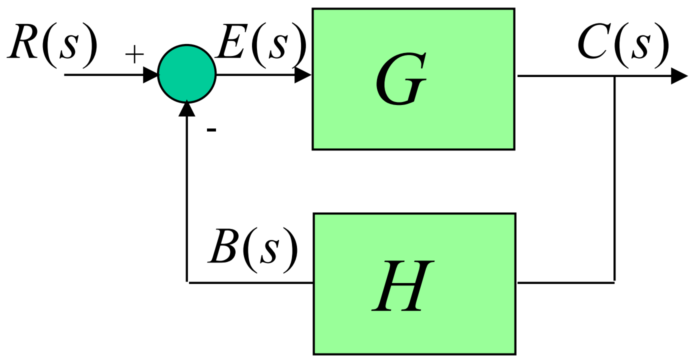
Parallel:



$$G = G_1 + G_2$$

Block Diagrams

Negative Feedback:



R	Reference input
$E \square R - B$	Error signal
$C \square GE$	Output
$B \square HC$	Feedback signal

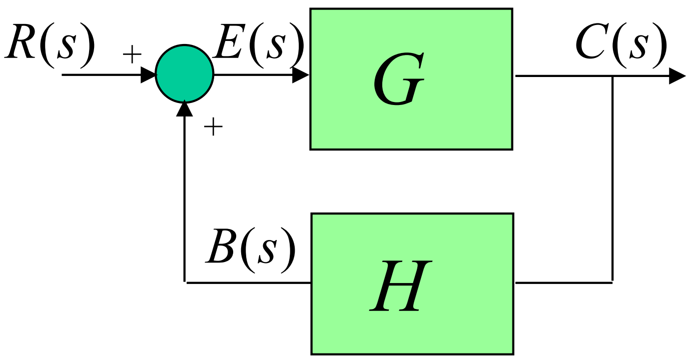
$$C \square GR - GHC \Rightarrow (1 \square GH)C \square GR \Rightarrow \frac{C}{R} \square \frac{G}{(1 \square GH)}$$

$$E \square R - HGE \Rightarrow (1 \square GH)E \square R \Rightarrow \frac{E}{R} \square \frac{1}{(1 \square GH)}$$

Rule: Transfer Function=Forward Gain/(1+Loop Gain)

Block Diagrams

Positive Feedback:



R	Reference input
$E = R + B$	Error signal
$C = GE$	Output
$B = HC$	Feedback signal

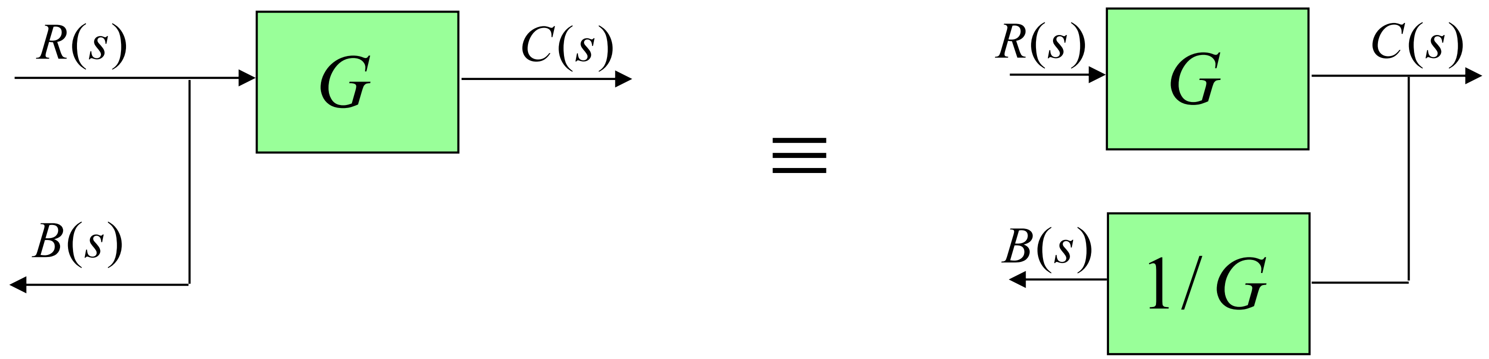
$$C = GR + GHC \Rightarrow (1 - GH)C = GR \Rightarrow \frac{C}{R} = \frac{G}{(1 - GH)}$$

$$E = R + HGE \Rightarrow (1 - GH)E = R \Rightarrow \frac{E}{R} = \frac{1}{(1 - GH)}$$

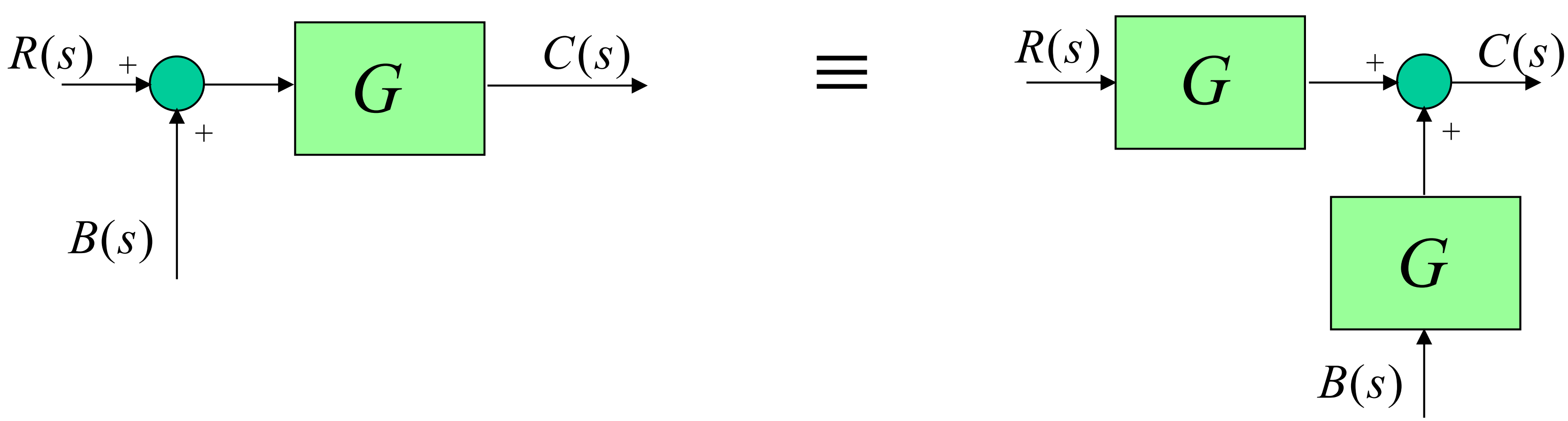
Rule: Transfer Function=Forward Gain/(1-Loop Gain)

Block Diagrams

Moving through a branching point:

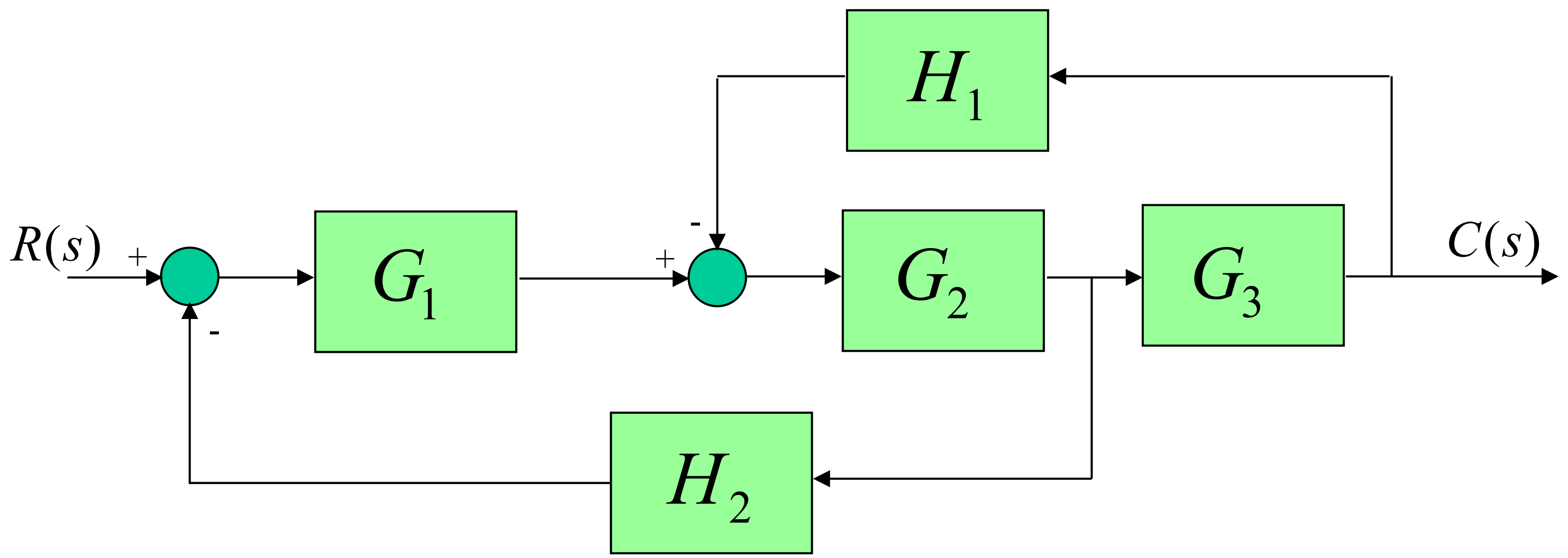


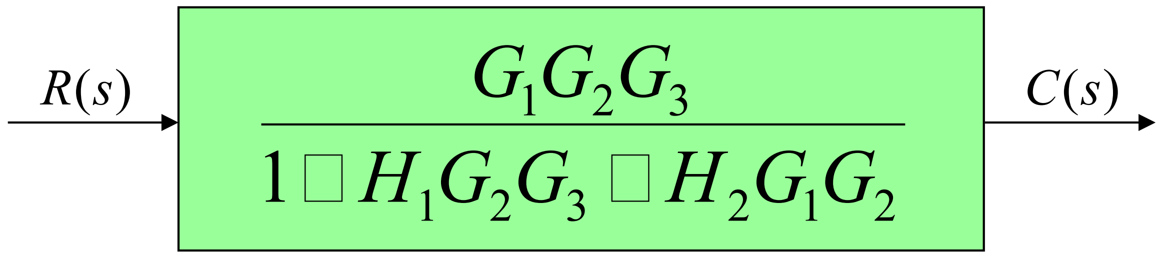
Moving through a summing point:



Block Diagrams

Example:





$$\frac{G_1 G_2 G_3}{1 \square H_1 G_2 G_3 \square H_2 G_1 G_2}$$

Impulse Response

Dirac's delta:

$$\int_0^\infty u(\tau)\delta(t-\tau)d\tau = u(t)$$

Integration is a limit of a sum

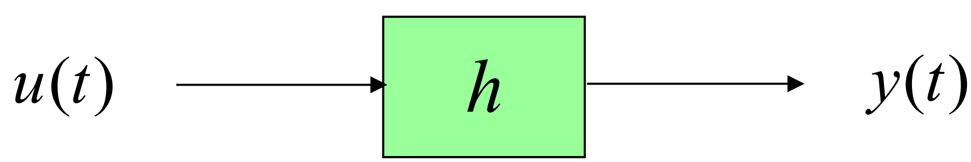
$$\Downarrow$$

$u(t)$ is represented as a sum of impulses

By superposition principle, we only need unit impulse response

$h(t-\tau)$ Response at t to an impulse applied at τ

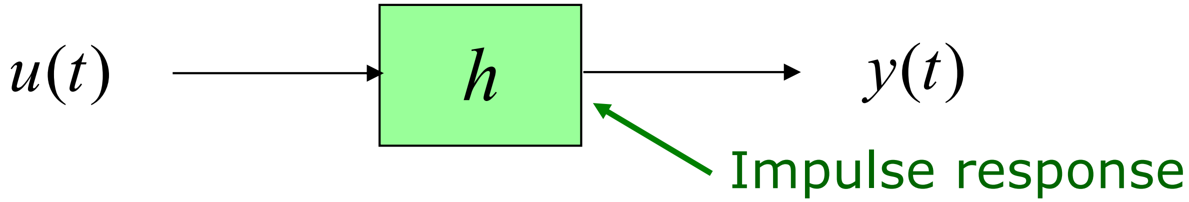
System Response:



$$y(t) = \int_0^\infty u(\tau)h(t-\tau)d\tau$$

Impulse Response

t-domain:

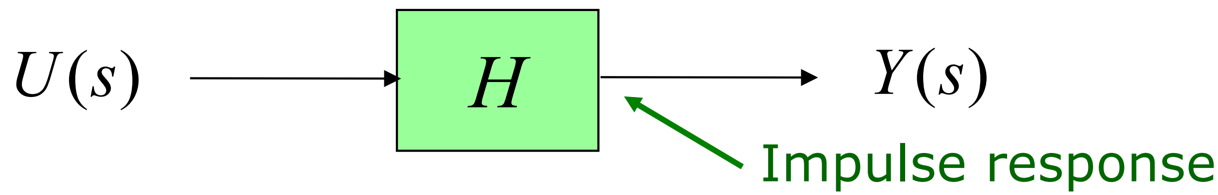


$$y(t) = \int_0^\infty u(\tau)h(t-\tau)d\tau \qquad u(t) = \delta(t) \Rightarrow y(t) = h(t)$$

The system response is obtained by convolving the input with the impulse response of the system.

Convolution: $\mathcal{L}\{\int_0^\infty u(\tau)h(t-\tau)d\tau\} = H(s)U(s)$

s-domain:



$$Y(s) = H(s)U(s) \qquad u(t) = \delta(t) \Rightarrow U(s) = 1 \Rightarrow Y(s) = H(s)$$

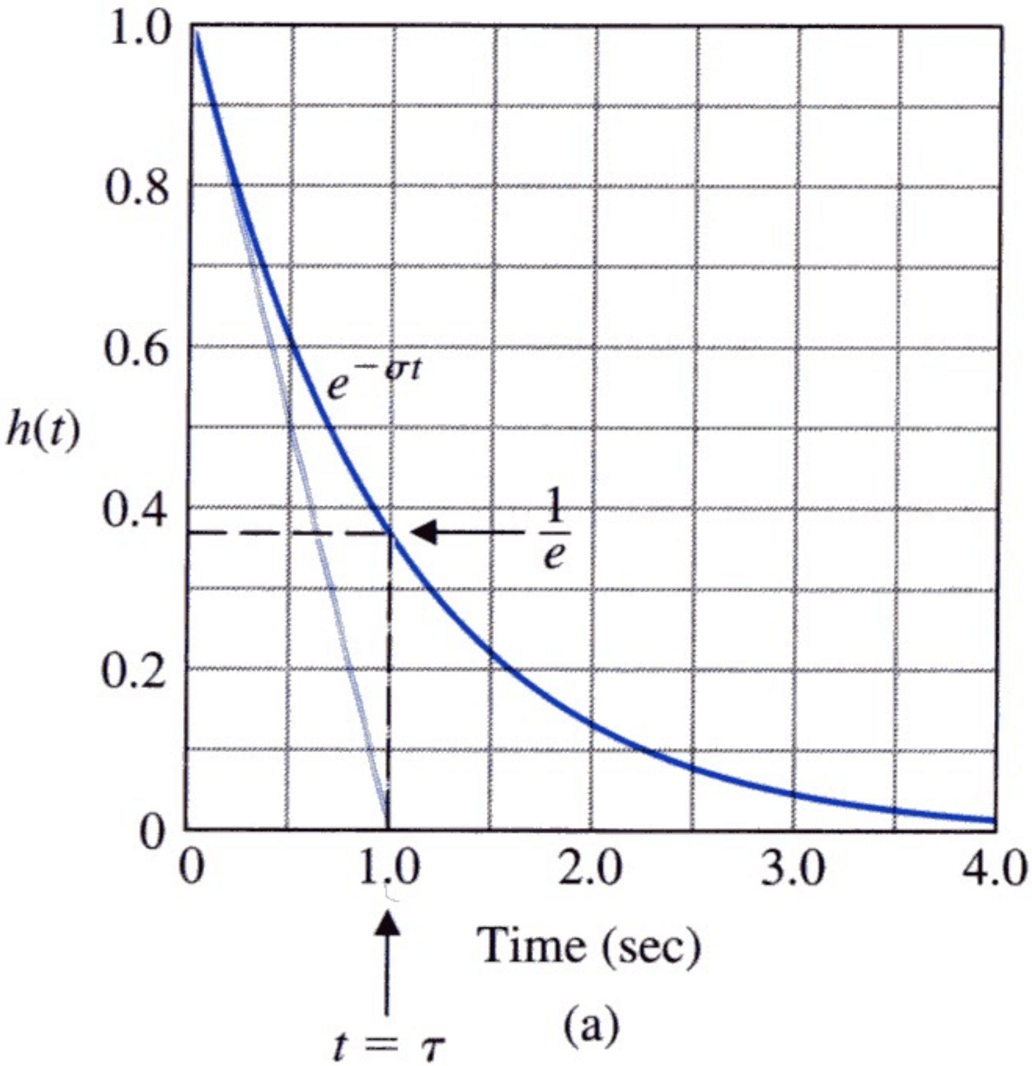
The system response is obtained by multiplying the transfer function and the Laplace transform of the input.

Time Response vs. Poles

Real pole:

$$H(s) = \frac{1}{s+\sigma} \Rightarrow h(t) = e^{-\sigma t}$$

Impulse Response



$\sigma > 0$ Stable

$\sigma < 0$ Unstable

$\tau = \frac{1}{\sigma}$ Time Constant

Time Response vs. Poles

Real pole:

$$H(s) = \frac{\sigma}{s+\sigma} \Rightarrow h(t) = \sigma e^{-\sigma t}$$

Impulse Response

$$\tau = \frac{1}{\sigma}$$

Time Constant

$$Y(s) = \frac{\sigma}{s+\sigma}\frac{1}{s} \Rightarrow y(t) = 1 - e^{-\sigma t}$$

Step Response

Example: ME207_TimeResponse_1.m

Time Response vs. Poles

Complex poles:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

Impulse Response

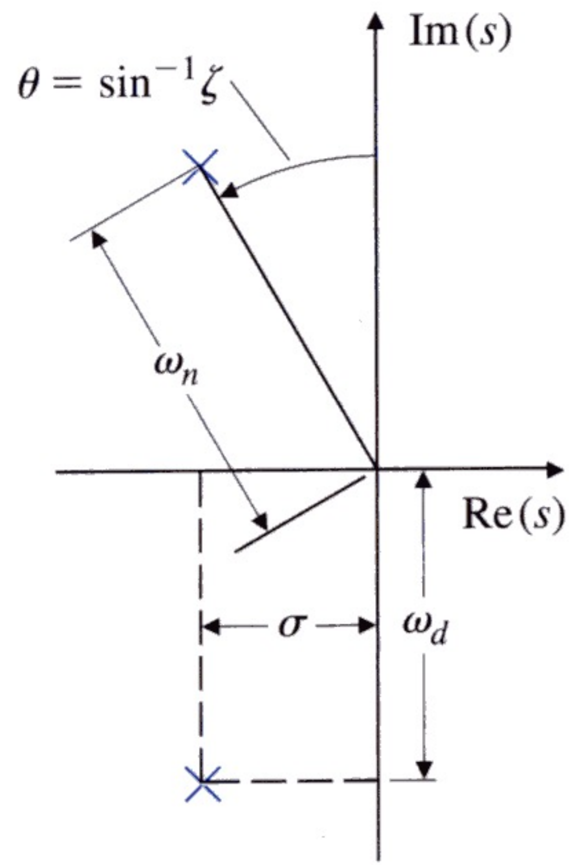
ω_n : Undamped natural frequency

ζ : Damping ratio

$$H(s) = \frac{\omega_n^2}{\left(s + \sigma + j\omega_d\right)\left(s + \sigma - j\omega_d\right)}$$

$$= \frac{\omega_n^2}{\left(s + \sigma\right)^2 + \omega_d^2}$$

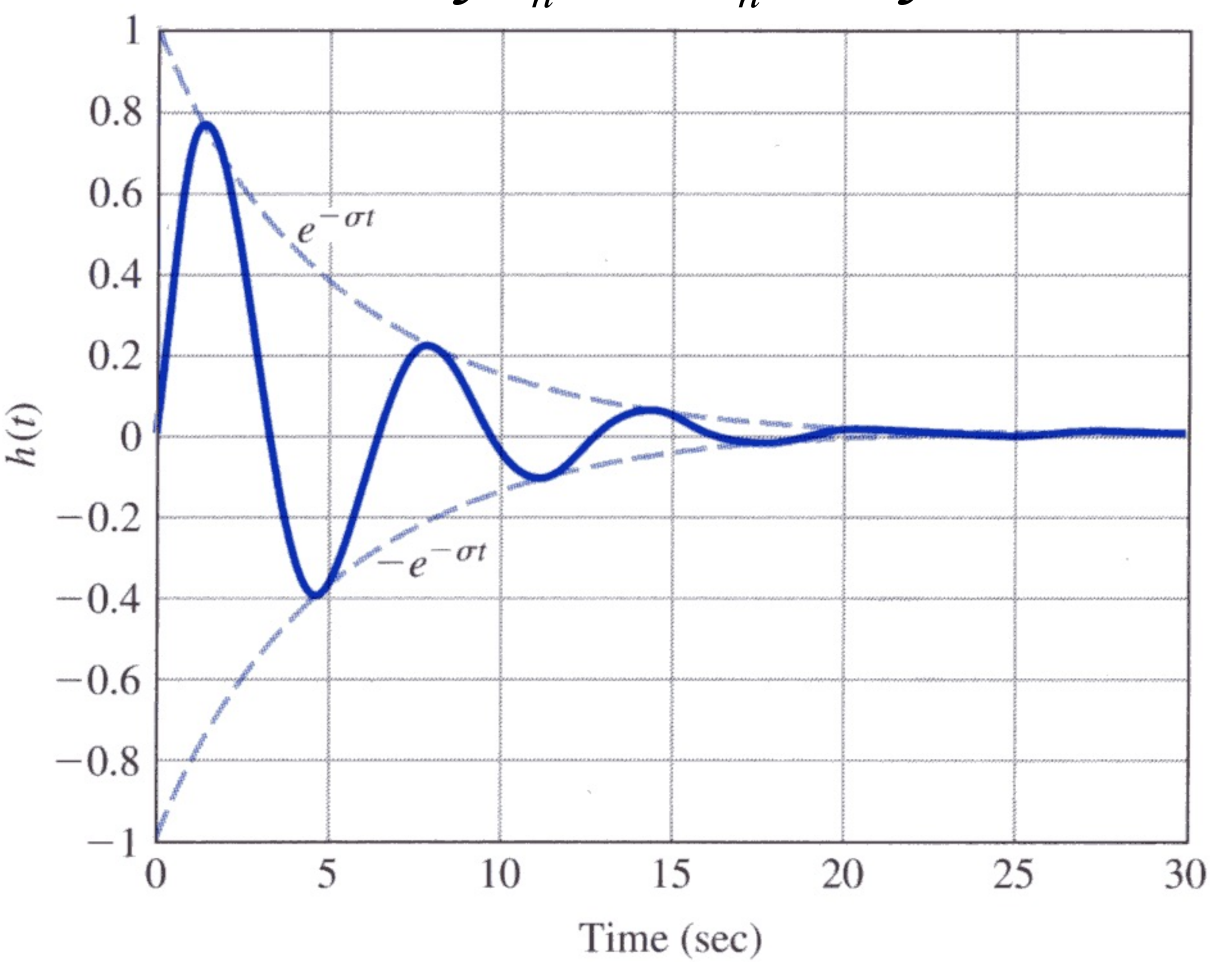
$$\sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1 - \zeta^2}$$



Time Response vs. Poles

Complex poles:

$$H(s) = \frac{\omega_n^2}{\left[(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)\right]} \rightarrow h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t)$$



Impulse Response

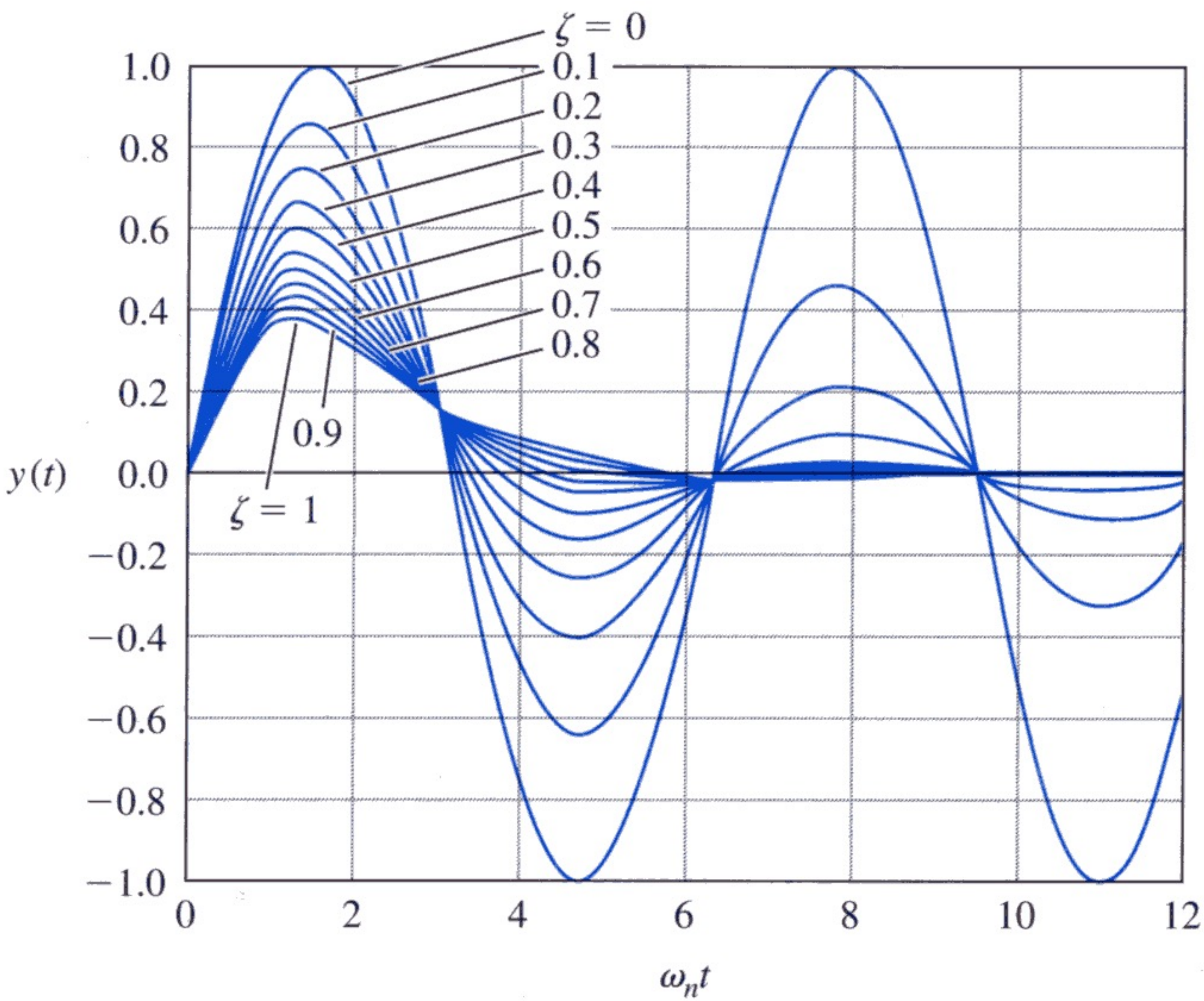
$\sigma > 0$ Stable

$\sigma < 0$ Unstable

Time Response vs. Poles

Complex poles:

$$H(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1-\zeta^2\right)} \rightarrow h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin\left(\omega_d t\right)$$

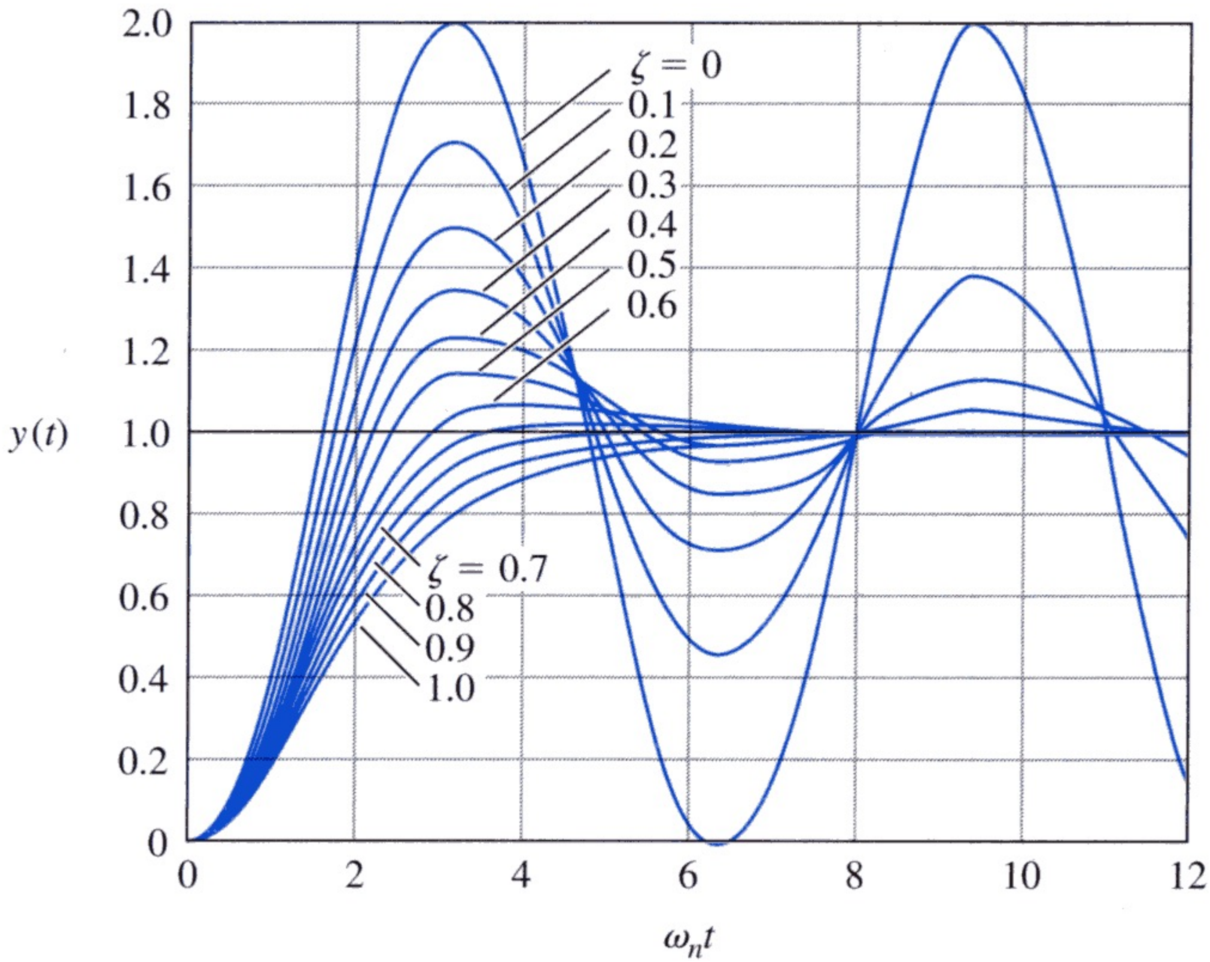


Impulse Response

Time Response vs. Poles

Complex poles:

$$Y(s) = \frac{\omega_n^2}{\left[(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)\right]}\frac{1}{s} \rightarrow y(t) = 1-e^{-\sigma t}\left[\cos(\omega_d t)+\frac{\sigma}{\omega_d}\sin(\omega_d t)\right]$$



Step Response

Time Response vs. Poles

Complex poles: $$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

CASES:

$\zeta = 0: s^2 = -\omega_n^2$ Undamped

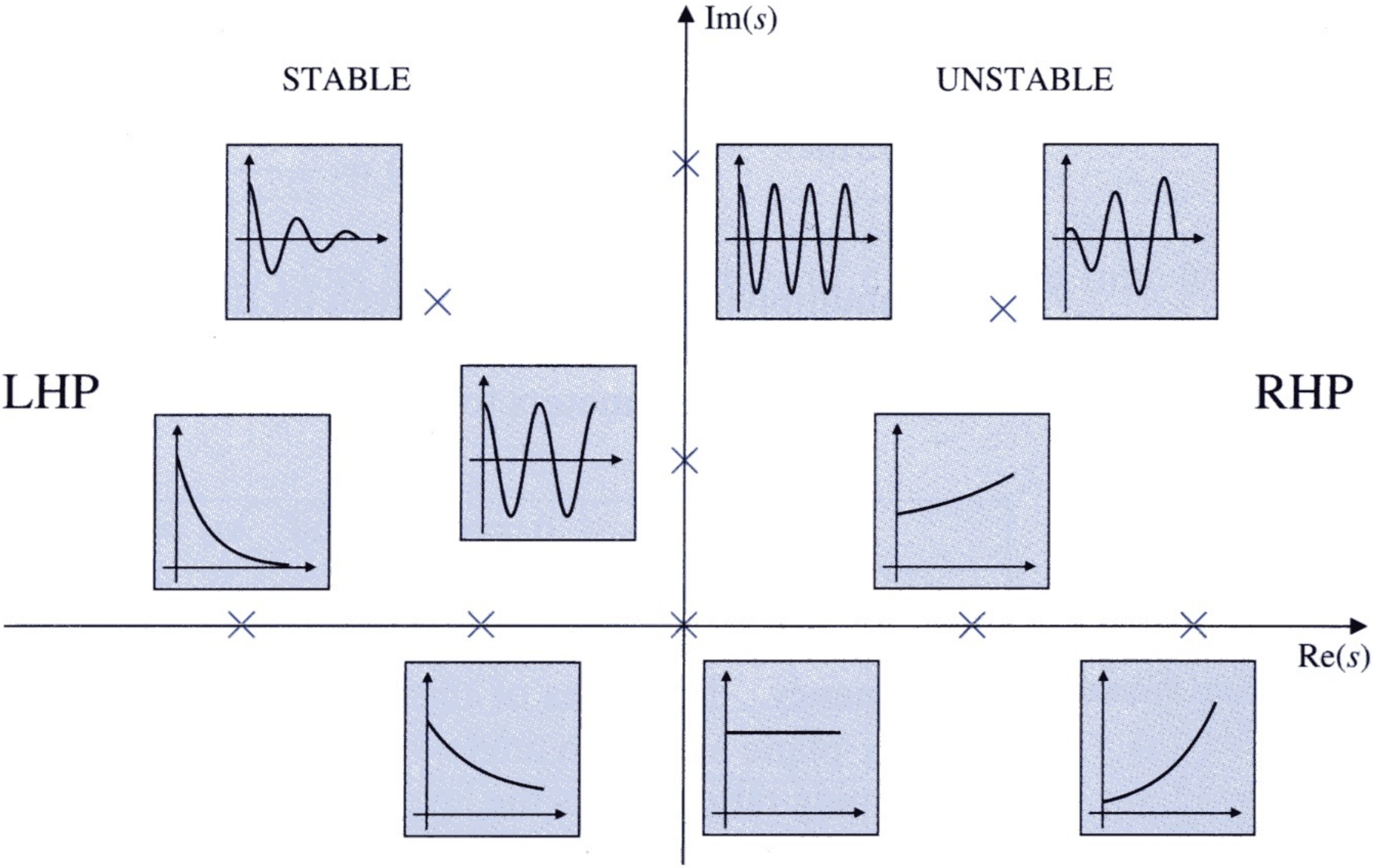
$\zeta < 1: \left(s + \zeta\omega_n\right)^2 = -\omega_n^2\left(1 - \zeta^2\right)$ Underdamped

$\zeta = 1: \left(s + \omega_n\right)^2$ Critically damped

$\zeta > 1: \left[s + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n\right]\left[s + \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n\right]$ Overdamped

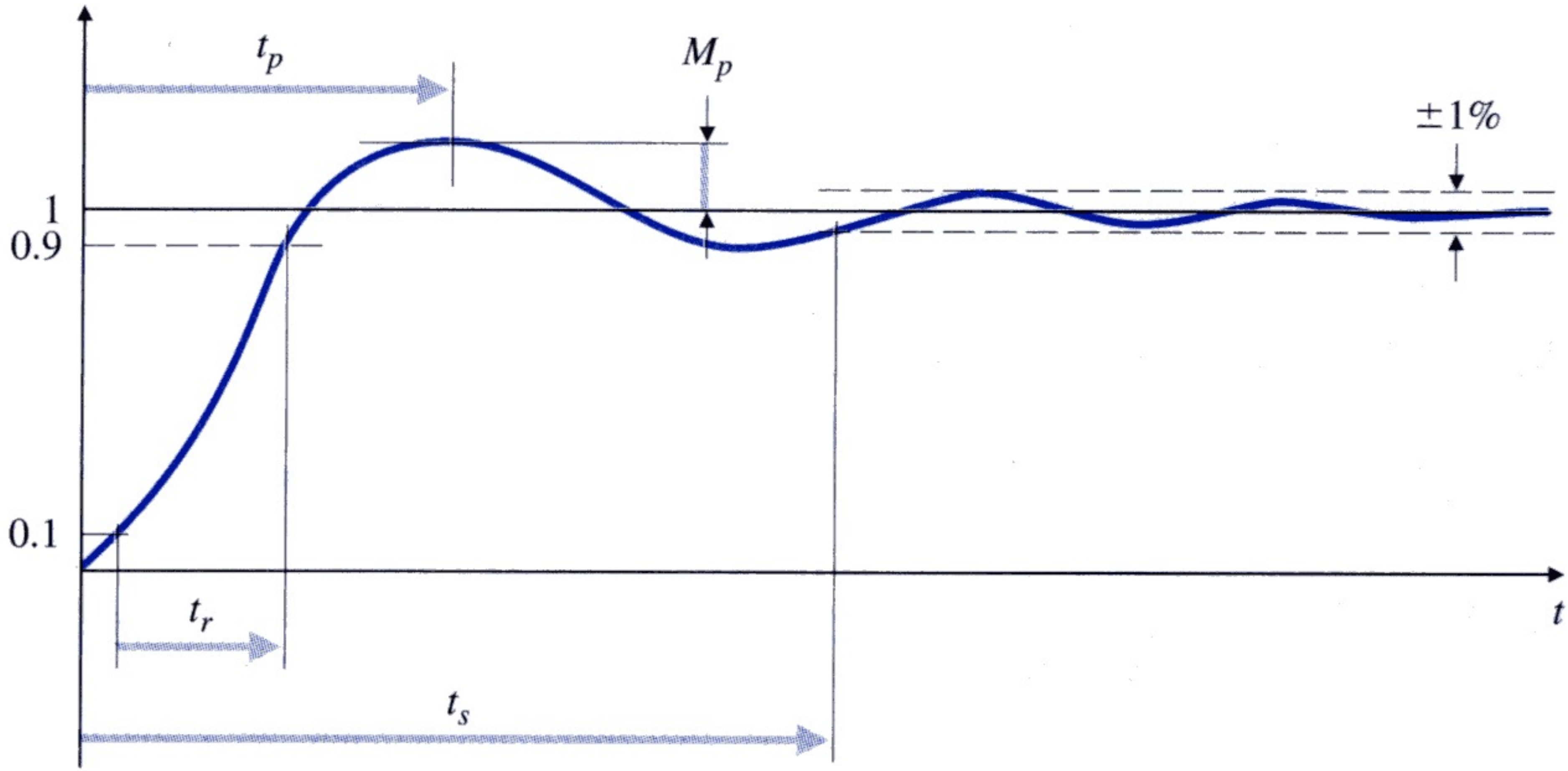
Example: ME207_TimeResponse_2.m

Time Response vs. Poles



Example: ME207_TimeResponse_Multiple.m

Time Domain Specifications



Time Domain Specifications

1- The **rise time** t_r is the time it takes the system to reach the vicinity of its new set point

2- The **settling time** t_s is the time it takes the system transients to decay

3- The **overshoot** M_p is the maximum amount the system overshoot its final value divided by its final value

4- The **peak time** t_p is the time it takes the system to reach the maximum overshoot point

$$t_p \approx \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad t_r \cong \frac{1.8}{\omega_n}$$

$$M_p \approx e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}} \qquad t_s \approx \frac{4.6}{\zeta\omega_n}$$

Time Domain Specifications

Design specification are given in terms of

$$t_r, t_p, M_p, t_s$$

These specifications give the position of the poles

$$\omega_n, \zeta \Rightarrow \sigma, \omega_d$$

Example: Find the pole positions that guarantee

$$t_r \leq 0.6\sec, M_p \,\square\, 10\%, t_s \leq 3\sec$$

$$t_r \leq 0.6s \Leftrightarrow \omega_n \geq \frac{1.8}{0.6s} = \frac{3}{s}$$

$$M_p \leq 0.1 \Leftrightarrow \zeta \geq \sqrt{\frac{(\ln 0.1)^2}{\pi^2 + (\ln 0.1)^2}} = 0.5912$$

$$t_s \leq 3s \Leftrightarrow \omega_n \geq \frac{4.6}{\zeta 3s} = \frac{1.5333}{\zeta s}$$

$$\zeta = 0.6 \Leftrightarrow \omega_n \geq \frac{4.6}{\zeta 3s} = \frac{2.5556}{s} \Rightarrow \omega_n = 3$$

Example: ME207_TimeSpecifications.m

Effect of Zeros and Additional poles

Additional poles:

1- can be neglected if they are sufficiently to the left of the dominant ones.

2- can increase the rise time if the extra pole is within a factor of 4 of the real part of the complex poles.

Example: ME207_TimeResponse_ExtraPole.m

Effect of Zeros and Additional poles

Zeros:

1- a zero near a pole reduces the effect of that pole in the time response.

2- a zero in the LHP will increase the overshoot if the zero is within a factor of 4 of the real part of the complex poles (due to differentiation).

3- a zero in the RHP (nonminimum phase zero) will depress the overshoot and may cause the step response to start out in the wrong direction.

Example: ME207_TimeResponse_ExtraZero.m

Stability

$$\frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$\frac{Y(s)}{R(s)} = K \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

$$\frac{Y(s)}{R(s)} = \frac{k_1}{(s - p_1)} + \frac{k_2}{(s - p_2)} + \cdots + \frac{k_n}{(s - p_n)}$$

Impulse response:

$$R(s) = 1 \Rightarrow Y(s) = \frac{k_1}{(s - p_1)} + \frac{k_2}{(s - p_2)} + \cdots + \frac{k_n}{(s - p_n)}$$

$$y(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_n e^{p_n t}$$

(See Laplace Transform table)

Stability

$$y(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_n e^{p_n t}$$

We want: $$e^{p_i t} \xrightarrow[t\to\infty]{} 0 \qquad \forall i = 1\ldots n$$

Definition: A system is asymptotically stable (a.s.) if

$$\mathrm{Re}\{p_i\} < 0 \qquad \forall i$$

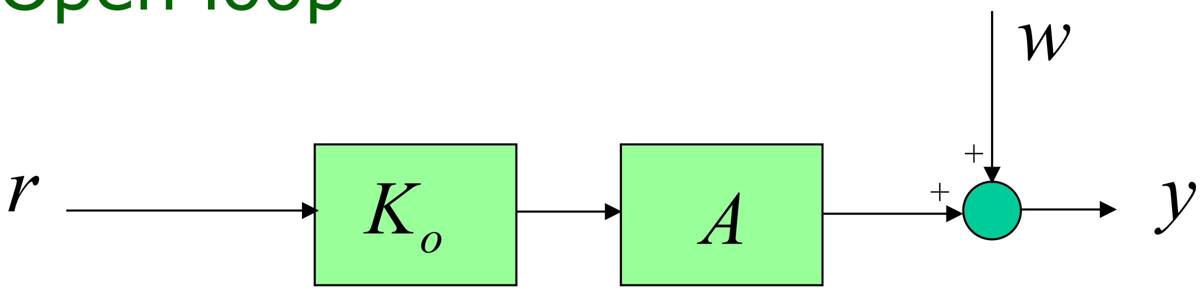
Characteristic polynomial: $$a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

Characteristic equation: $$a(s) = 0$$

Properties of feedback

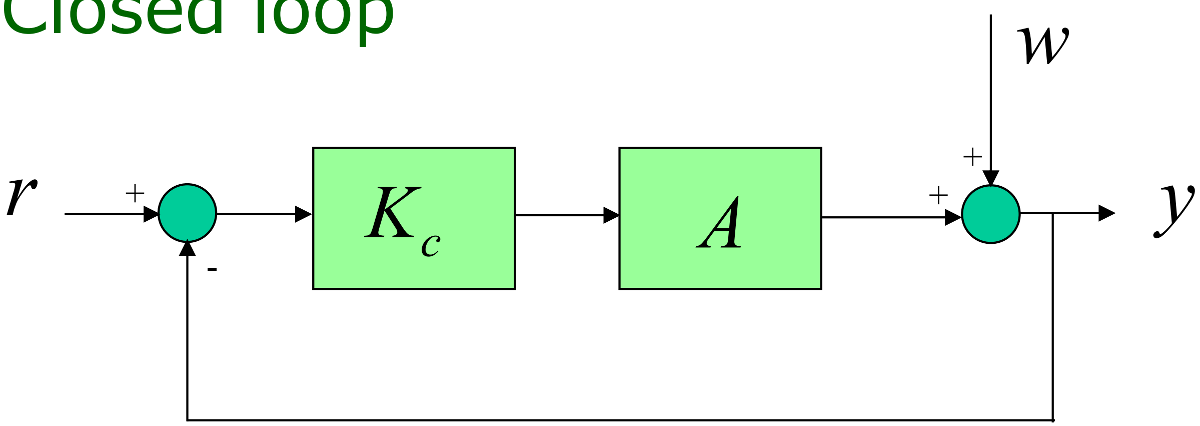
Disturbance Rejection:

Open loop



$$y = K_o A r + w$$

Closed loop



$$y = \frac{K_c A}{1 + K_c A} r + \frac{1}{1 + K_c A} w$$

Properties of feedback

Disturbance Rejection:

Choose control s.t. for $w=0, y \approx r$

Open loop: $$K_o = \frac{1}{A} \Rightarrow y = r + w$$

Closed loop: $$K_c \gg \frac{1}{A} \Rightarrow y \approx r + 0w = r$$

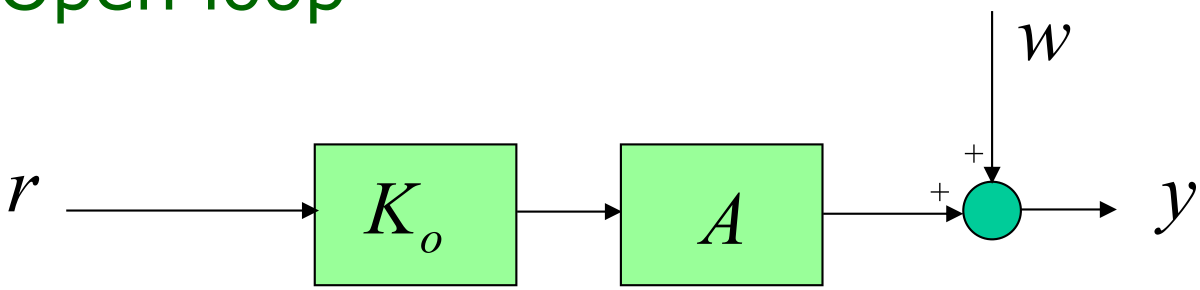
Feedback allows attenuation of disturbance without having access to it (without measuring it)!!!

IMPORTANT: High gain is dangerous for dynamic response!!!

Properties of feedback

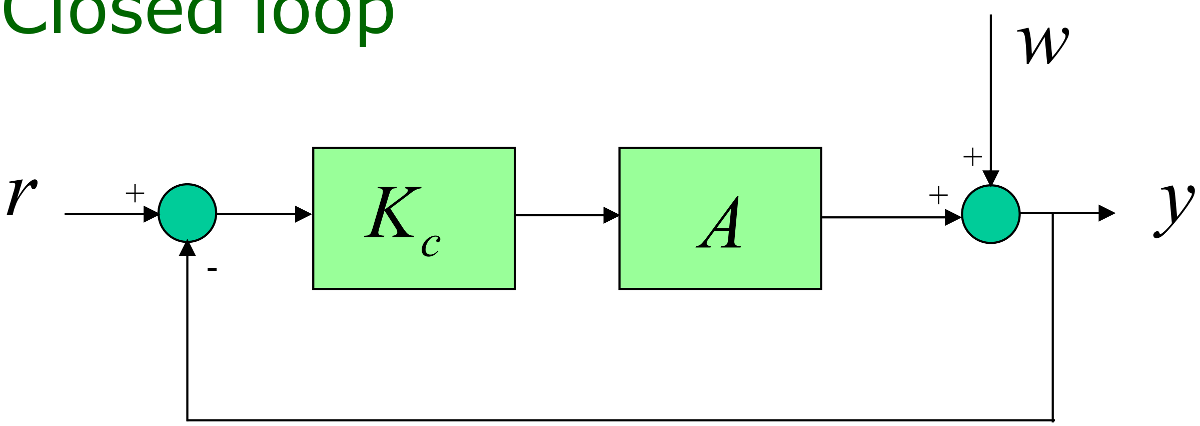
Sensitivity to Gain Plant Changes

Open loop



$$T_o = \left(\frac{y}{r}\right)_o = AK_o$$

Closed loop



$$T_c = \left(\frac{y}{r}\right)_c = \frac{AK_c}{1 + AK_c}$$

Properties of feedback

Sensitivity to Gain Plant Changes

Let the plant gain be $A \pm \delta A$

Open loop: $$\frac{\delta T_o}{T_o} = \frac{\delta A}{A}$$

Closed loop: $$\frac{\delta T_c}{T_c} = \frac{\delta A}{A}\frac{1}{1+AK_c} \ll \frac{\delta A}{A} = \frac{\delta T_o}{T_o}$$

Feedback reduces sensitivity to plant variations!!!

Sensitivity: $$S_A^T = \frac{dT/T}{dA/A} = \frac{A}{T}\frac{dT}{dA}$$

Example: $$S_A^{T_c} = \frac{1}{1+AK_c}, S_A^{T_o} = 1$$

Simulation of feedback systems via Simulink

Example: ME207_TimeResponse_cldesign_setup.m
ME207_TimeResponse_cldesign.mdl
ME207_TimeResponse_cldesign_run.m

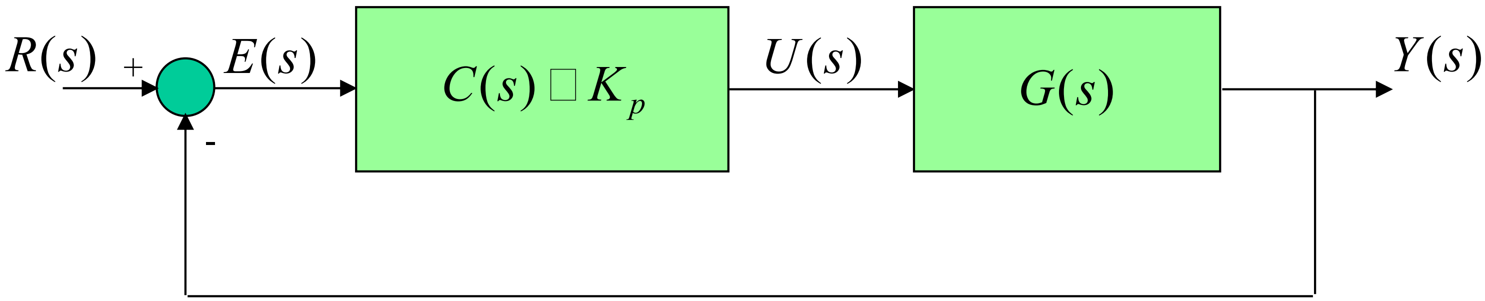
PID Controller

PID: Proportional – Integral – Derivative

P Controller:

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)},$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + C(s)G(s)}.$$



$$u(t) = K_p e(t), \qquad U(s) = K_p E(s)$$

Step Reference:

$$R(s) = \frac{1}{s} \Rightarrow e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + K_p G(s)} \frac{1}{s} = \frac{1}{1 + K_p G(0)}$$

$$e_{ss} = 0 \Leftrightarrow K_p G(0) \to \infty$$

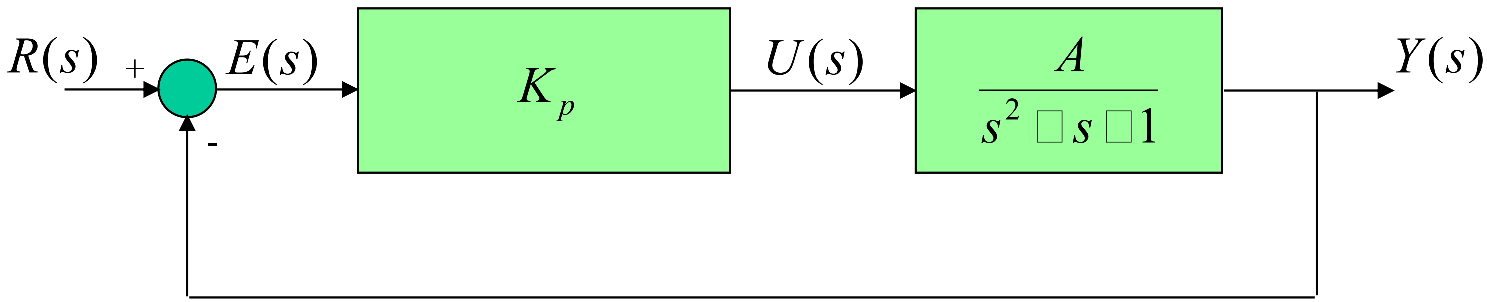
True when:

- Proportional gain is high
- Plant has a pole at the origin

High gain proportional feedback (needed for good tracking) results in underdamped (or even unstable) transients.

PID Controller

P Controller: Example (ME207_P_Controller.m)



$$\frac{Y(s)}{R(s)} = \frac{K_p G(s)}{1 + K_p G(s)} = \frac{K_p A}{s^2 + s + (1 + K_p A)}$$

$$\omega_n^2 = 1 + K_p A$$

$$2\zeta\omega_n = 1$$

$$\Rightarrow \zeta = \frac{1}{2\omega_n} = \frac{1}{2\sqrt{1 + K_p A}} \xrightarrow[K_p \to \infty]{} 0$$

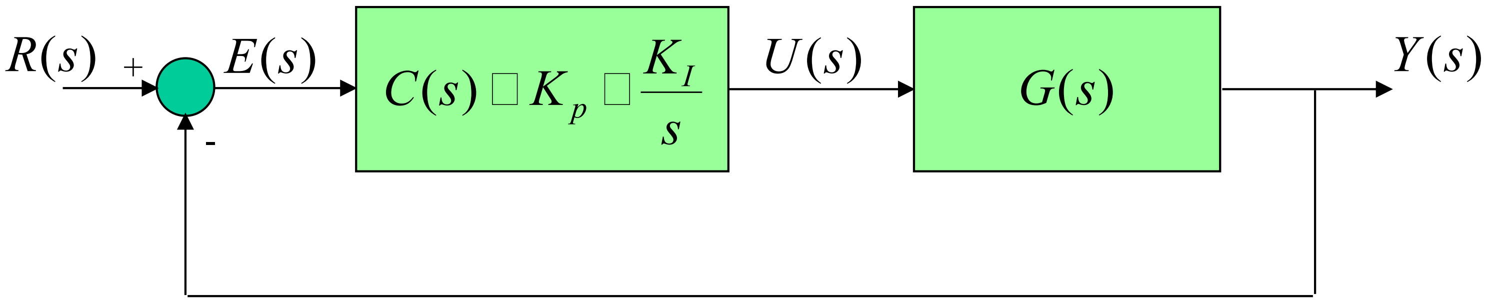
- ✓ Underdamped transient for large proportional gain
- ✓ Steady state error for small proportional gain

PID Controller

PI Controller:

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)},$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+C(s)G(s)}.$$



$$u(t) = K_p e(t) + K_I \int_0^t e(\tau)d\tau, \qquad U(s) = \left(K_p + \frac{K_I}{s}\right)E(s)$$

Step Reference:

$$R(s) = \frac{1}{s} \Rightarrow e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{1}{1+\left(K_p + \frac{K_I}{s}\right)G(s)} \frac{1}{s} = \lim_{s\to 0} \frac{1}{1+\left(K_p + \frac{K_I}{s}\right)G(s)} = 0$$

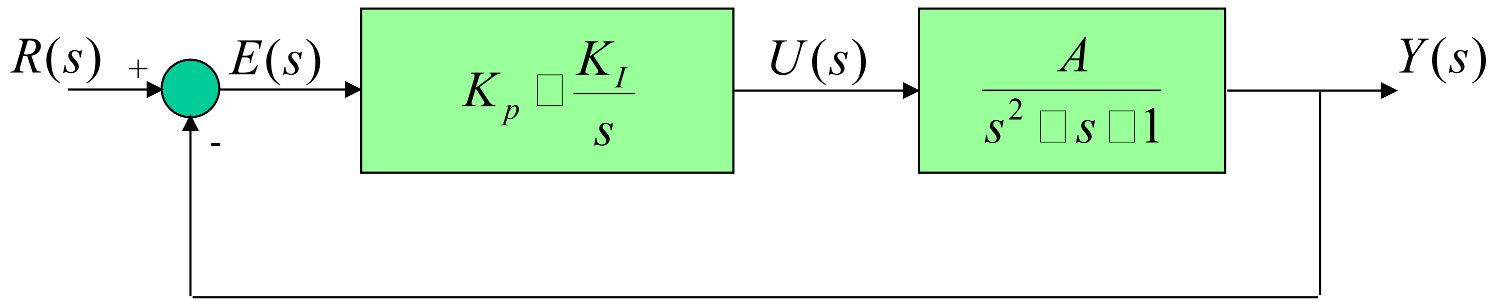
- It does not matter the value of the proportional gain
- Plant does not need to have a pole at the origin. The controller has it!

Integral control achieves perfect steady state reference tracking!!!

Note that this is valid even for $K_p=0$ as long as $K_i \neq 0$

PID Controller

PI Controller: Example (ME207_PI_Controller.m)



$$\frac{Y(s)}{R(s)} = \frac{\left(K_p + \frac{K_I}{s}\right)G(s)}{1 + \left(K_p + \frac{K_I}{s}\right)G(s)} = \frac{\left(K_p s + K_I\right)A}{s^3 + s^2 + (1 + K_p A)s + K_I A}$$

DANGER: for large K_i the characteristic equation has roots in the RHP

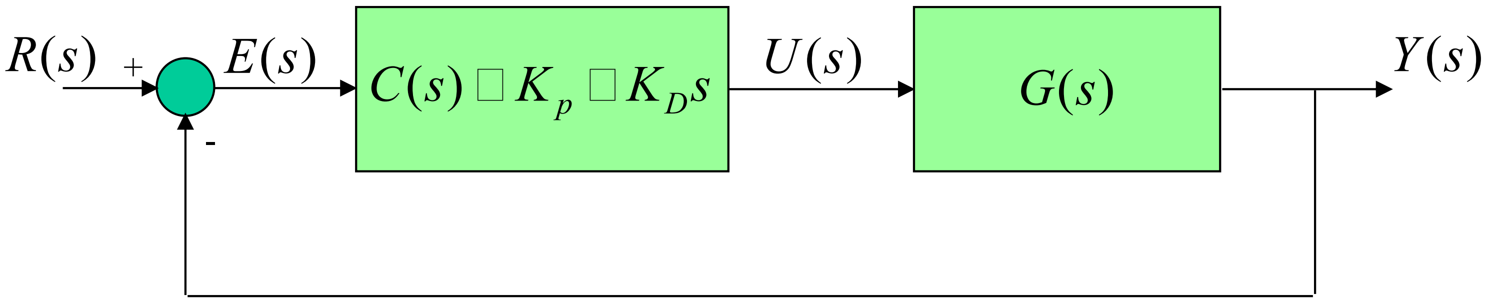
$$s^3 + s^2 + (1 + K_p A)s + K_I A = 0$$

PID Controller

PD Controller:

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)},$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+C(s)G(s)}.$$



$$u(t) = K_p e(t) + K_D \frac{de(t)}{dt}, \qquad U(s) = \left(K_p + K_D s\right)E(s)$$

Step Reference:

$$R(s) = \frac{1}{s} \Rightarrow e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{1}{1+\left(K_p + K_D s\right)G(s)} \frac{1}{s} = \frac{1}{1+K_p G(0)}$$

$$e_{ss} = 0 \Leftrightarrow K_p G(0) \to \infty$$

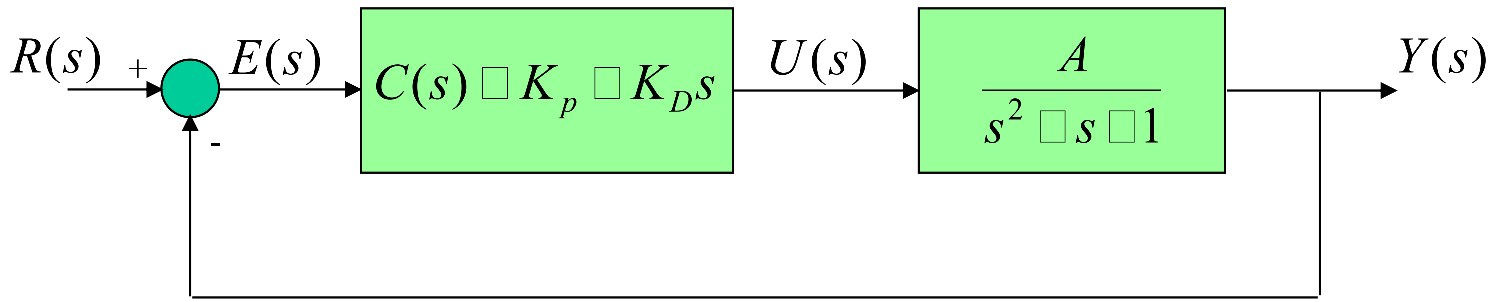
True when:

- Proportional gain is high
- Plant has a pole at the origin

PD controller fixes problems with stability and damping by adding "anticipative" action

PID Controller

PD Controller: Example (ME207_PD_Controller.m)



$$\frac{Y(s)}{R(s)} = \frac{\left(K_p + K_D s\right)G(s)}{1 + \left(K_p + K_D s\right)G(s)} = \frac{A\left(K_p + K_D s\right)}{s^2 + \left(1 + K_D A\right)s + (1 + K_p A)}$$

$$\omega_n^2 = 1 + K_p A$$

$$2\zeta\omega_n = 1 + K_D A \qquad \Rightarrow \zeta = \frac{1 + K_D A}{2\omega_n} = \frac{1 + K_D A}{2\sqrt{1 + K_p A}}$$

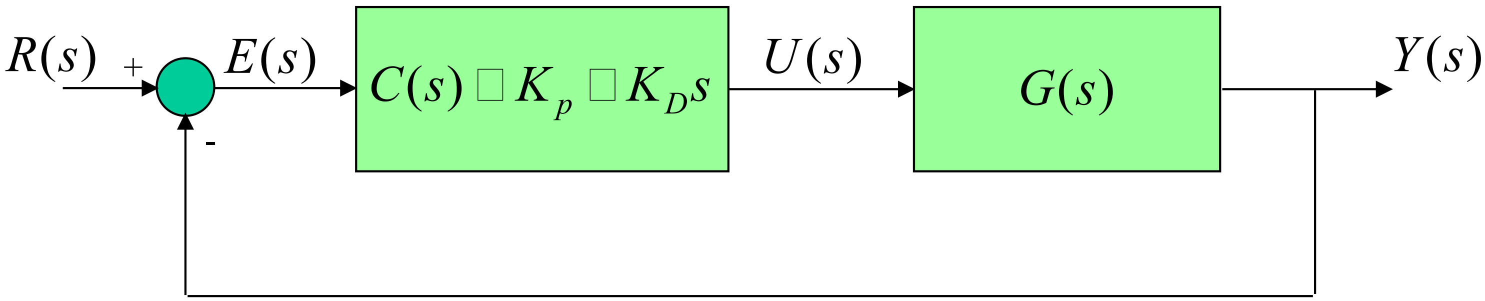
✓ The damping can be increased now independently of K_p

✓ The steady state error can be minimized by a large K_p

PID Controller

PD Controller:

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)},$$
$$\frac{E(s)}{R(s)} = \frac{1}{1+C(s)G(s)}.$$



$$u(t) = K_p e(t) + K_D \frac{de(t)}{dt}, \qquad U(s) = (K_p + K_D s)E(s)$$

NOTE: cannot apply pure differentiation.
In practice,

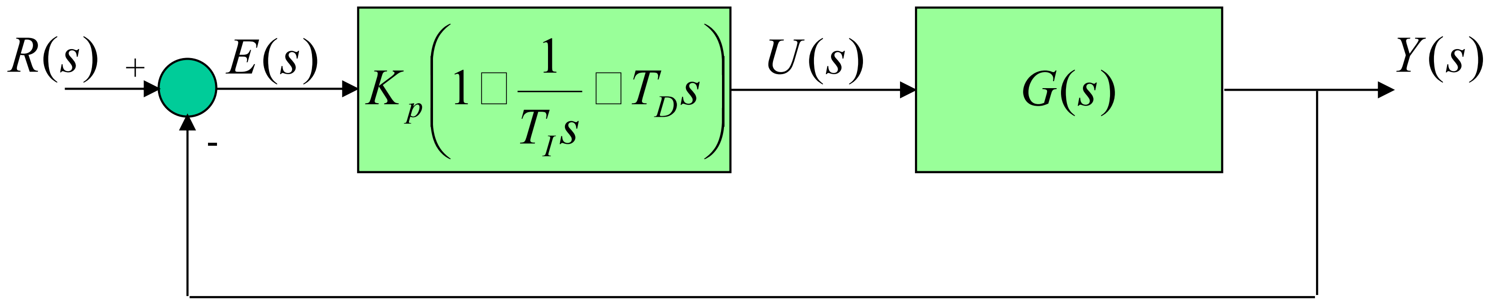
$$K_D s$$

is implemented as

$$\frac{K_D s}{\tau_D s + 1}$$

PID Controller

PID: Proportional – Integral – Derivative



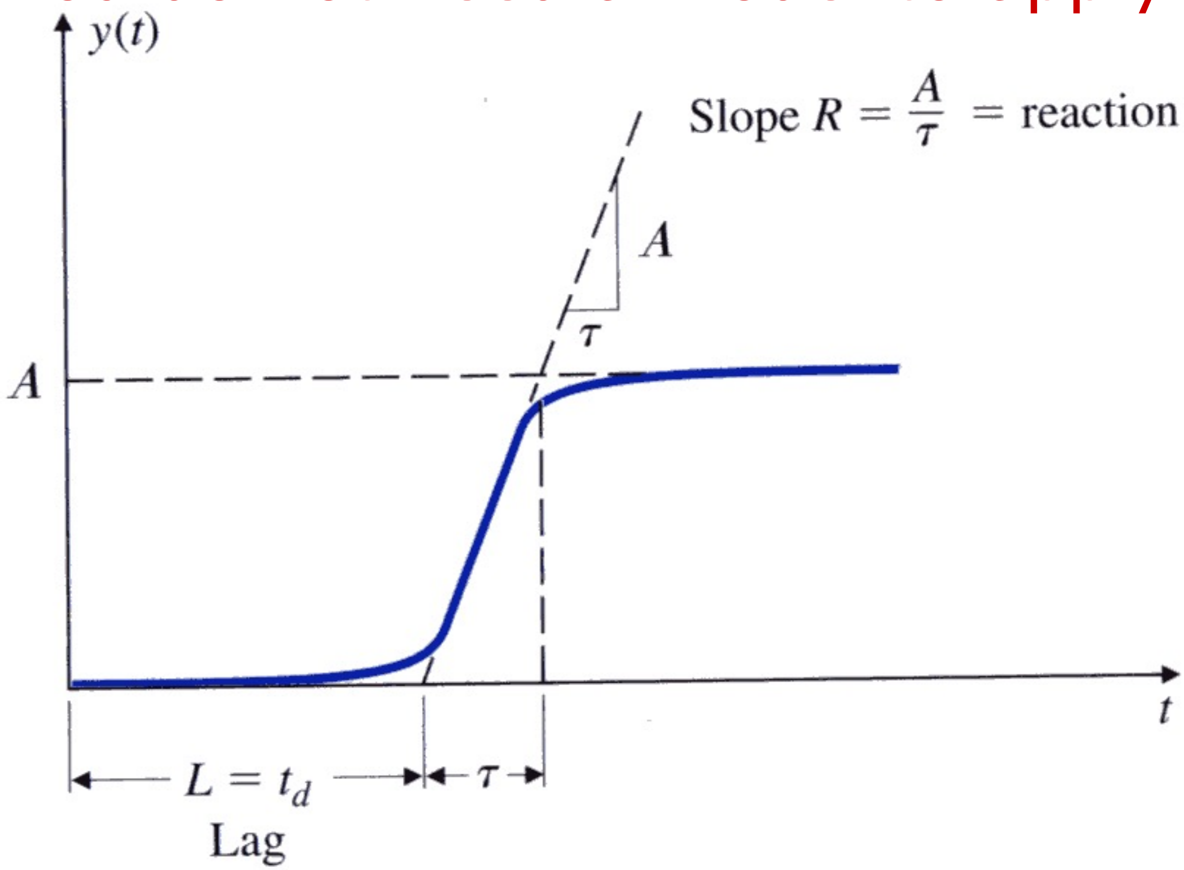
$$u(t) = K_p\left[e(t) + \frac{1}{T_I}\int_0^d e(\tau)d\tau + T_D\frac{de(t)}{dt}\right] \quad K_I = \frac{K_p}{T_I}, K_D = K_p T_D$$

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_I s} + T_D s\right)$$

PID Controller: Example (ME207_PID_Controller.m)

PID Controller: Ziegler-Nichols Tuning

- Empirical method (no proof that it works well but it works well for simple systems)
- Only for stable plants
- You do not need a model to apply the method

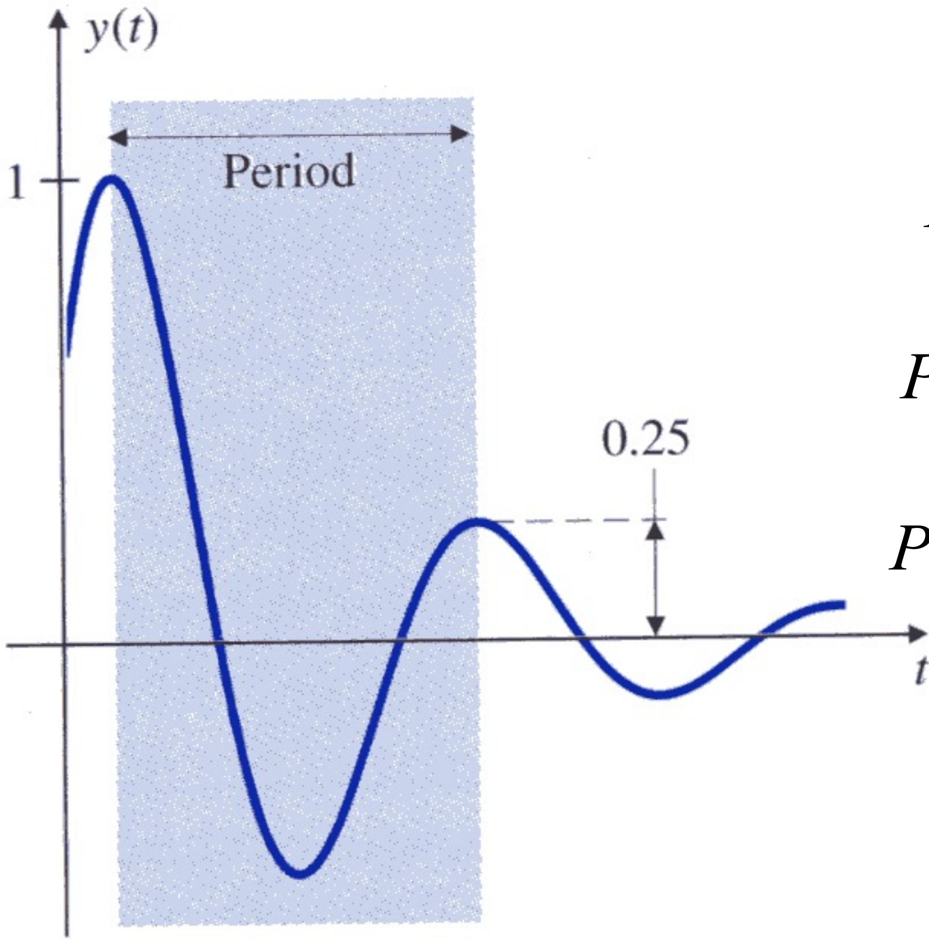


Class of plants:

$$\frac{Y(s)}{U(s)} = \frac{Ke^{-t_d s}}{\tau s + 1}$$

PID Controller: Ziegler-Nichols Tuning

METHOD 1: Based on step response, tuning to decay ratio of 0.25.



Tuning Table:

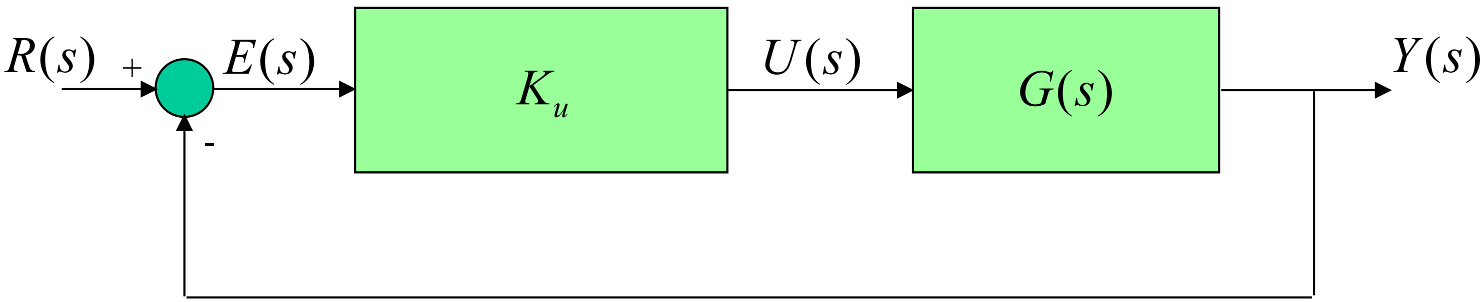
$$P: \quad K_p \square \frac{\tau}{t_d}$$

$$PD: \quad K_p \square 0.9\frac{\tau}{t_d}, T_I \square \frac{t_d}{0.3}$$

$$PID: \quad K_p \square 1.2\frac{\tau}{t_d}, T_I \square 2t_d, T_D \square 0.5t_d$$

PID Controller: Ziegler-Nichols Tuning

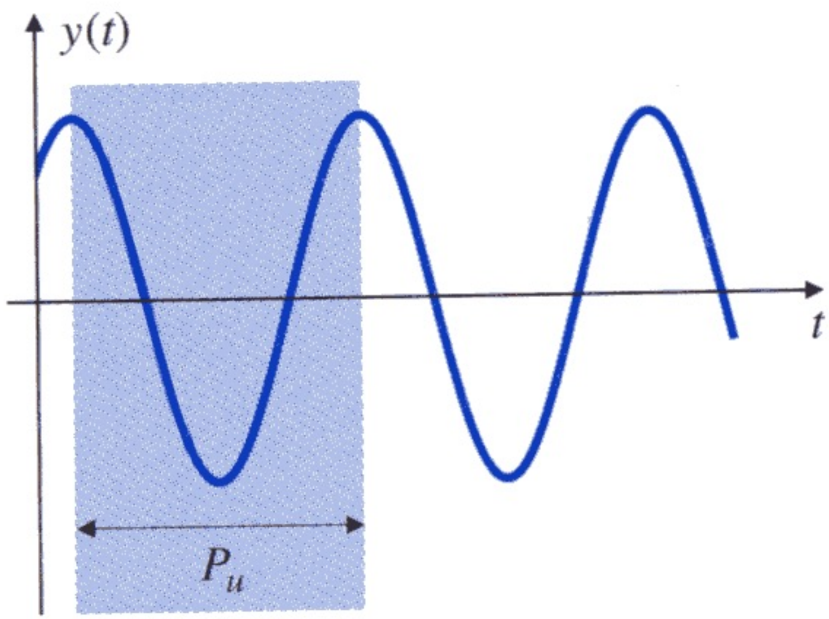
METHOD 2: Based on limit of stability, ultimate sensitivity method.



- Increase the constant gain K_u until the response becomes purely oscillatory (no decay – marginally stable – pure imaginary poles)
- Measure the period of oscillation P_u

PID Controller: Ziegler-Nichols Tuning

METHOD 2: Based on limit of stability, ultimate sensitivity method.



Tuning Table:

$$P: \quad K_p \square 0.5K_u$$

$$PD: \quad K_p \square 0.45K_u, T_I \square \frac{P_u}{1.2}$$

$$PID: \quad K_p \square 0.6K_u, T_I \square \frac{P_u}{2}, T_D \square \frac{P_u}{8}$$

The Tuning Tables are the same if you make:

$$K_u \square 2\frac{\tau}{t_d}, P_u \square 4t_d$$