Score Sequences for Tournaments

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Score Sequences of Round Robin Tournaments

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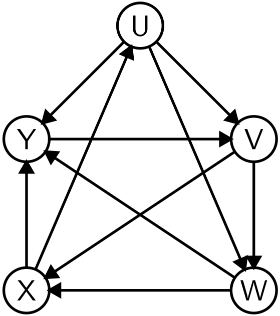
U wins 3 games, V wins 2 games, W wins 2 games, X wins 2 games, Y wins 1 games

Score sequence is (3,2,2,2,1)



Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

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Try testing *ALL* possible tournaments?

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Use mathematical tools to make the check faster

For 7 players there are $\dfrac{7(7-1)}{2} = 21$ games in a round robin tournament

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Which of the following are score sequences for a tournament with 7 players?

$(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$

$(5, 4, 3, 3, 3, 1, 0)$

$(3, 3, 3, 3, 3, 3, 3)$

$(6, 6, 4, 2, 1, 1, 1)$

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$(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$ NO - Scores must be non-negative integers

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$(3, 3, 3, 3, 3, 3, 3)$

$(6, 6, 4, 2, 1, 1, 1)$ NO - Last 5 teams must have at least $10 = \frac{5 \cdot 4}{2}$ wins

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$(3, 3, 3, 3, 3, 3, 3)$ YES

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Landau (1951) considered tournaments in the context of pecking order in poultry populations

A necessary condition for a sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers to be the score sequence of a round-robin tournament:

$$\sum_{i \in I} s_i \geq \frac{|I|(|I|-1)}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\}$$

$$\text{with equality when } I = \{1, 2, \ldots, n\}$$

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The number of wins for any set of teams must be as large as the number of games played between those teams
and
the total number of wins must equal the total number of games played

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Landau's Theorem: these necessary conditions are also sufficient

If the conditions hold there is a tournament with the given score sequence

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If the conditions hold there is a tournament with the given score sequence

If not a score sequence then there is a set of teams violating these obvious conditions

The sequence

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

can be checked by hand in a few minutes. It is not a score sequence

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$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Not a score sequence

Last 10 teams have 44 wins in $45 = \frac{10 \cdot 9}{2}$ games

Landau's Theorem:
A sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers is a score sequence of a round-robin tournament if and only if

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What if we allow ties?

What if the score is 3 points for a win, 1 for a tie and 0 for a loss (world cup soccer scoring)?

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What is $\binom{|I|}{2}$?

$$\binom{13}{2} = \frac{13 \cdot 12}{2} = 8 \text{ choose } 2$$

$$=$$

Number of 2 element subsets of $\{1, 2, \ldots, 13\}$

Binomial coefficients $\binom{n}{k} = n$ choose k

$$=$$

number of k elements subsets of $\{1, 2, \ldots, n\}$

$$\binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2}$$

$$\binom{13}{5} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2}$$

Binomial coefficients - ‘Pascal’s Triangle’

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

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1							
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1	7	21	35	35	21	7	1

Hayluda Bhattotpala (India around 1000)
Al-Karaji and Kayyam (Persia around 1050)
Yang Hui (China around 1350)
Tartaglia (Italy around 1550)
Pascal (France around 1650)

Binomial identity: $\binom{7}{3} = \binom{6}{2} + \binom{6}{3}$

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

"Proof":

The $\binom{7}{3}=35$ size 3 subsets of $\{A, B, C, D, E, F, G\}$

$=$

The $\binom{6}{2} = 15$ subsets including A $+$ The $\binom{6}{3} = 20$ subsets avoiding A

$$\begin{array}{lllllllll}
 & 1 \\
 & 1 & 1 \\
 & 1 & 2 & 1 \\
\color{red}{8 =} & \color{red}{1} & \color{red}{3} & \color{red}{3} & \color{red}{1} \\
 & 1 & 4 & 6 & 4 & 1 \\
 & 1 & 5 & 10 & 10 & 5 & 1 \\
 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
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\end{array}$$

$$
\begin{array}{rcllllllll}
 & & 1 & & & & & & & \\
 & & 1 & 1 & & & & & & \\
 & & 1 & 2 & 1 & & & & & \\
\color{red}{8} & \color{red}{=} & \color{red}{1} & \color{red}{3} & \color{red}{3} & \color{red}{1} & & & & \\
\color{blue}{16} & \color{blue}{=} & \color{blue}{1} & \color{blue}{4} & \color{blue}{6} & \color{blue}{4} & \color{blue}{1} & & & \\
 & & 1 & 5 & 10 & 10 & 5 & 1 & & \\
 & & 1 & 6 & 15 & 20 & 15 & 6 & 1 & \\
 & & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
$$

$$
\begin{array}{rclllllll}
 & & 1 \\
 & & 1 & 1 \\
 & & 1 & 2 & 1 \\
8 & = & 1 & 3 & 3 & 1 \\
16 & = & 1 & 4 & 6 & 4 & 1 \\
32 & = & 1 & 5 & 10 & 10 & 5 & 1 \\
 & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
 & & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
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\begin{array}{rcl}
1 & = & 1 \\
2 & = & 1\ 1 \\
4 & = & 1\ 2\ 1 \\
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16 & = & 1\ 4\ 6\ 4\ 1 \\
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Row sums are powers of 2

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Row sums are powers of 2

"Proof": $128 = 2^7$, number of subsets of $\{1, 2, \ldots, 7\}$
row sums over choices of subset size

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
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1							
1	1						
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Diagonal sums are binomial coefficients:

$1 + 3 + 6 + 10 + 15 = 35$

Diagonal sums are binomial coefficients:

"Proof":

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
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\begin{array}{rllllllll}
 & 1 \\
 & 1 & 1 \\
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3 = & 1 & 3 & 3 & 1 \\
 & 1 & 4 & 6 & 4 & 1 \\
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Anti-diagonal sums are Fibonacci numbers

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$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

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$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0, F_1 = 1.$

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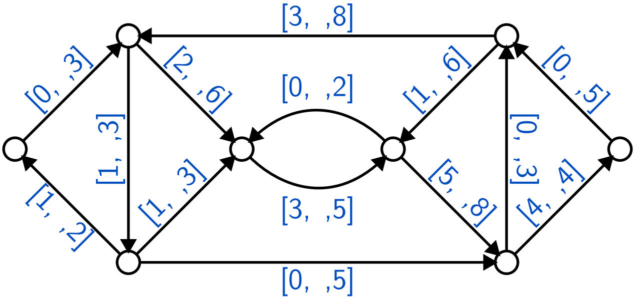
$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0, F_1 = 1.$

"Proof": Use binomial identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Circulation in a network: flow between lower and upper bounds satisfying flow conservation at each node



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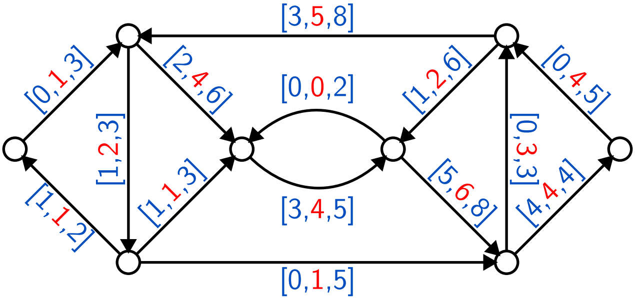

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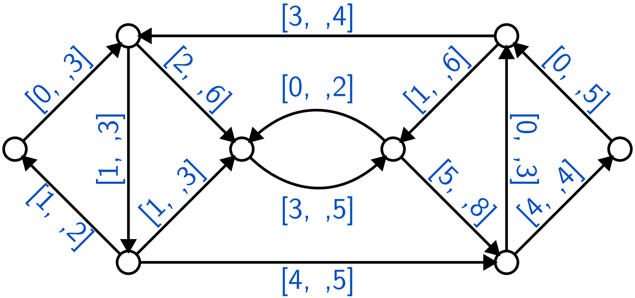
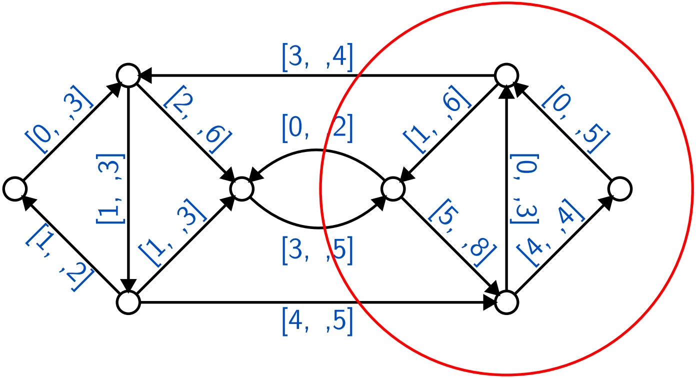
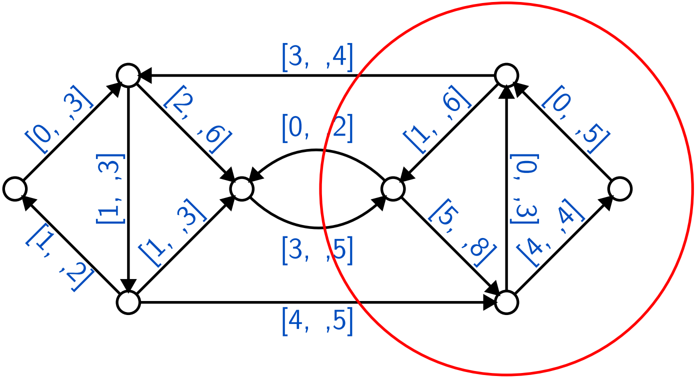




Requirements in $= 3 + 4 = 7 > 6 = 4 + 2 =$ capacity out



Requirements in $= 3 + 4 = 7 > 6 = 4 + 2 =$ capacity out

No circulation

Hoffman (1956)

A necessary condition for a circulation:

for for every set of nodes:

capacities out $\geq$ the requirements in

(sum of upper bounds) $\geq$ (sum of lower bounds in)

Hoffman (1956)

A necessary condition for a circulation:
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capacities out $\geq$ the requirements in
(sum of upper bounds) $\geq$ (sum of lower bounds in)

Hoffman's Circulation Theorem (1956): These necessary conditions are also sufficient

If the conditions hold there is a circulation

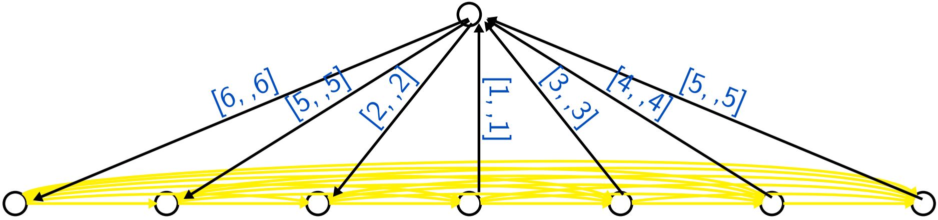
Hoffman (1956)

A necessary condition for a circulation:
for for every set of nodes:
capacities out $\geq$ the requirements in
(sum of upper bounds) $\geq$ (sum of lower bounds in)

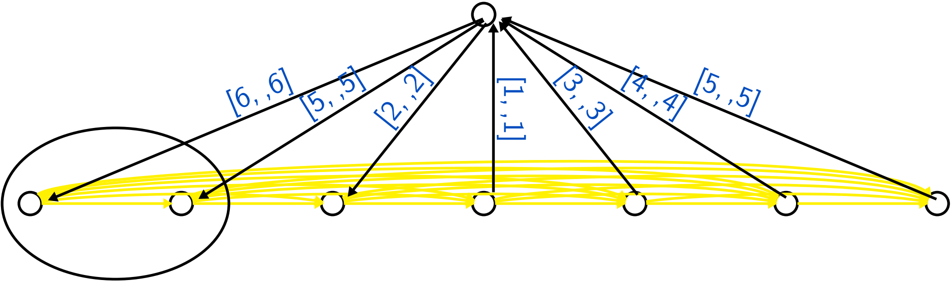
Hoffman's Circulation Theorem (1956): These necessary conditions are also sufficient

If the conditions hold there is a circulation

If there is no circulation the is some set with capacities out $<$ requirements in



Yellow arcs left to right, lower bound 0, upper bound 1



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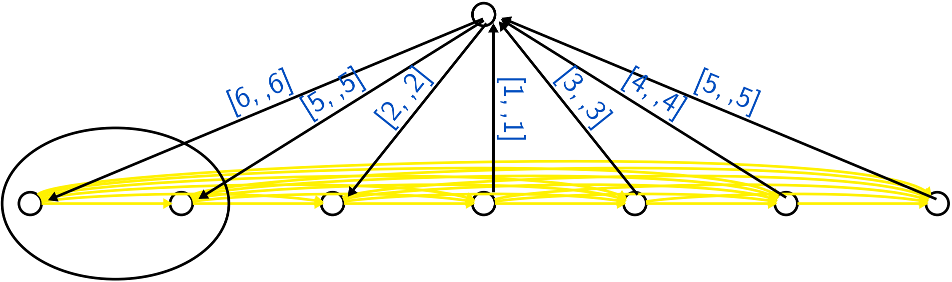
Requirements in $= 11 > 10 =$ capacities out



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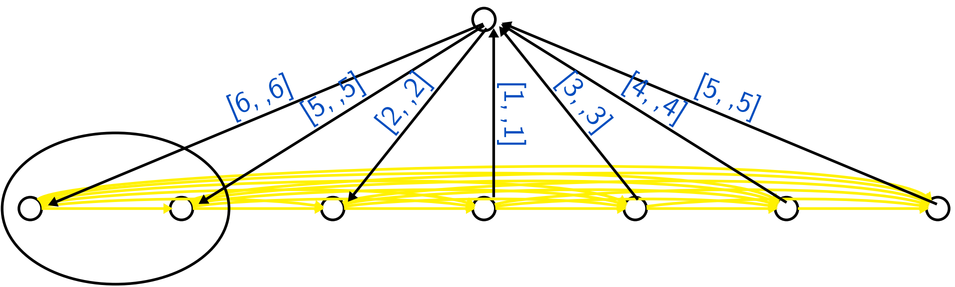
No circulation



Yellow arcs left to right, lower bound 0, upper bound 1
Requirements in $= 11 > 10 =$ capacities out

No circulation

Corresponds to $(6, 6, 4, 2, 1, 1, 1)$
Not a score sequence

Algebraic approach to circulations:

Create a variable to represent flow on each arc

Put variables between lower and upper bounds and satisfy flow conservation

Solve resulting system of linear equations and inequalities

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Farkas' Lemma for linear systems $\Rightarrow$ Hoffman's circulation theorem $\Rightarrow$ Landau's Theorem

Do these have nonnegative solutions?

$$\begin{aligned} x+2y &= 3 \\ 4x+5y &= 6 \end{aligned} \qquad \begin{aligned} x+2y &= 3 \\ 4x+8y &= 12 \end{aligned} \qquad \begin{aligned} x+2y &= 3 \\ 4x+8y &= 6 \end{aligned}$$

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$x=-1,\ y=2$	line $x+2y=3$	Has no solution
no	yes	Why not?

Do these have nonnegative solutions?

$$\begin{aligned} x + y + 2z &= 3 \\ 5x + 8y + 13z &= 21 \end{aligned} \qquad \begin{aligned} x + y + 2z &= 13 \\ 5x + 8y + 13z &= 21 \end{aligned}$$

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$x = 0$, $y = z = 1$

yes

no

Why not?

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Set up systems for circulations and score sequences. The ‘multipliers’ are $0, 1$ produce violations of necessary conditions.