Complete Classification of Tournaments Having a Disjoint Union of Directed Paths as a Minimum Feedback Arc Set

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Abstract: A feedback arc set of a digraph is a set of arcs whose reversal makes the resulting digraph acyclic. Given a tournament with a disjoint union of directed paths as a feedback arc set, we present necessary and sufficient conditions for this feedback arc set to have minimum size. We will present a construction for tournaments where the difference between the size of a minimum feedback arc set and the size of the largest collection of arc disjoint cycles can be made arbitrarily large. We will also make a

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connection to a problem found in [Barthélemy et al., 2]. The reversing number of a digraph was defined to be $r(D) = |V(T)| - |V(D)|$ where T is a smallest tournament having the arc set of D as a minimum feedback arc set. As a consequence of our classification of all tournaments having a disjoint union of directed paths as a minimum feedback arc set, we will obtain a new result involving the reversing number. We obtain precise reversing numbers for all digraphs consisting of a disjoint union of directed paths.

Keywords: *feedback arc sets; linear ordering problem; tournaments*

1. INTRODUCTION

A feedback arc set of a digraph is a set of arcs that when reversed (or deleted) makes the resulting digraph acyclic. A minimum feedback arc set is simply a smallest sized feedback arc set. Given a tournament that has a disjoint union of directed paths as a feedback arc set we establish necessary and sufficient conditions for this feedback arc set to have minimum size. We will show in Theorem 2.2 that if a tournament has a feedback arc set consisting of a disjoint union of directed paths, then this feedback arc set is minimum-sized if and only if the tournament does not contain one of the subdigraphs (not necessarily induced) shown in Figure 5 or 6. In Theorem 2.1 we will use this classification to construct tournaments where the gap between the size of a minimum feedback arc set and the size of the largest collection of arc disjoint cycles is approximately $\frac{7}{8}$ the number of vertices.

We will also consider the following problem posed in [2]. Given an acyclic digraph D, determine the size of a smallest tournament T that has the arc set of D as a minimum feedback arc set. It was shown in [2] that for any digraph D, there exists some such T, and the *reversing number of a digraph*, $r(D)$, was defined to be $|V(T)| - |V(D)|$, and $r(D)$ was computed for several families of digraphs. As a consequence of our classification of all tournaments having a disjoint union of directed paths as a feedback arc set, we will obtain a new result involving the reversing number of a digraph. In Theorem 3.1 we will determine precise reversing numbers for all digraphs consisting of a disjoint union of directed paths.

We will use $A(D)$ to denote the arc set of a digraph D, but to simplify matters when $A(D)$ is a feedback arc set of a tournament T, we will just say that "D is a feedback arc set of T." P_n will denote a directed path with n vertices and $\sum_{i=1}^{m} D_i$ will denote a disjoint union of m digraphs, sometimes referred to in other literature as the sum of m digraphs. If the $D_i's$ are all the same digraph we will simply use mD_i. For a subset of arcs $A(H) \subseteq A(D)$, the reversal of $A(H)$ denoted $(A(H))^R$, is defined to be the set $\{(y,x) \mid (x,y) \in A(H)\}$. Given an acyclic digraph D, if T is a tournament having D as a minimum feedback arc set, and no smaller tournament has D as a minimum feedback arc set, we will say that T *realizes* D, or that D has a *realizing tournament* T. For any other undefined notation, see [8].

Let D be a digraph and let W be a set of isolated "extra vertices," where $W \bigcap V(D) = \emptyset$. Let σ be an ordering of $\{V(D) \cup W\}$ that is *consistent* with D, meaning that, for each $(x, y) \in A(D), \sigma(y) < \sigma(x)$. Let $T(\sigma, D)$ be the tournament with vertex set $V(D) \bigcup W$ such that $(T(\sigma, D) \backslash A(D)) \cup (A(D))^R$ is acyclic. That is, $A(T(\sigma, D)) = \{(y, x) \mid \sigma(x) < \sigma(y) \text{ and } (y, x) \in A(D)\} \cup \{(x, y) \mid \sigma(x) < \sigma(y)$ and $(x, y) \notin A(D)\}$. We note that D is a feedback arc set of $T(\sigma, D)$. Moreover, when D is a feedback arc set of a tournament T, T must have this form for some ordering σ and some set W that is dependent on σ. To simplify notation we will say "Let σ be an ordering consistent with D" when we have a digraph D with an ordering σ of $V(D) \cup W$ where D is a feedback arc set of the tournament with vertex set equal to $V(D) \cup W$.

Our standard method for presenting a tournament $T(\sigma, D)$ will be in a diagram such as Figure 1, where the ordering σ arranges the vertices from left to right. Hence the arcs of D are directed from right to left and all other arcs are directed from left to right. In this setting, we can define an inverse map $\sigma^{-1} : \mathcal{Z}^+ \to V(T)$ and refer to the ith vertex in this ordering with $\sigma^{-1}(i)$.

The problem of finding a minimum feedback arc set in a tournament can be thought of in the context involving the ranking of players in a tournament. Let the vertices of the tournament correspond to players in a round robin tournament and there is an arc from vertex x to vertex y if and only if the player corresponding to vertex x beats the player corresponding to vertex y. Finding a minimum feedback arc set is then equivalent to finding a ranking with the minimum number of *inconsistencies*, where one player beats another and the loser is ranked higher than the winner. Furthermore, we note that the ranking of the players is simply the ordering of the acyclic tournament created upon reversal of these inconsistent arcs. We formally define the term inconsistencies below.

Definition 1.1. *Let T be a tournament and let π be a permutation of the labels of the vertices of T. The set of inconsistencies denoted INC (π, T), is the set* $\{(x, y) \in A(T) \mid \pi(x) > \pi(y)\}$. *Given a tournament T, an ordering π is said to be*

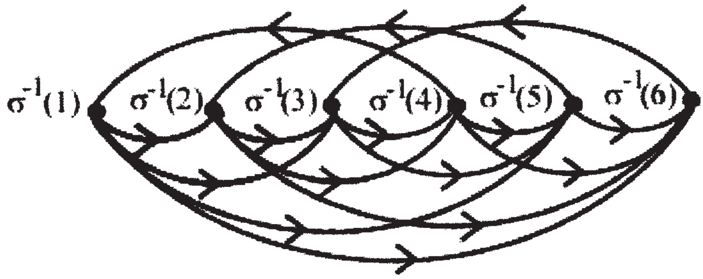

FIGURE 1. $T(\sigma, D)$ with $D = 3P_2$ as a feedback arc set.

optimal if for any ordering π' of $V(T)$, $|INC(\pi, T)| \leq |INC(\pi', T)|$ and an ordering π is said to be strictly better than π' if $|INC(\pi, T)| < |INC(\pi', T)|$.

In the context of rankings the problem is as follows. For a given set of inconsistencies, determine the size of a smallest tournament, which has this set as a minimum set of inconsistencies. The reversing number is then the fewest number of extra players that must be added to create a tournament with the prescribed minimum set of inconsistencies.

The study of minimum feedback arc sets was introduced in [7] and [9]. These sets are also referred to reversing sets in [2]. They are related to other sets studied in electrical engineering, statistics, and mathematics. We next state a well-known elementary result as our Lemma 1.1.

Lemma 1.1. *Let D be a feedback arc set of T. If T contains a collection of $|A(D)|$ arc disjoint cycles then D is a minimum feedback arc set of T.*

It is easy to see why this lemma is true. If T contains a collection of $|A(D)|$ arc disjoint cycles, then any feedback arc set would have to include at least one arc from each of these cycles. Hence any feedback arc set is at least as large as $|A(D)|$, so it follows that the arcs of D must form a minimum feedback arc set.

We note that the converse of Lemma 1.1 is not true in general and has been well studied [3]. In [3], probabilistic methods were used to show the existence of tournaments with a minimum feedback arc set of size $\frac{1}{2}\binom{n}{2} - cn^{\frac{3}{2}}$. The size of a largest collection of arc disjoint cycles is bounded above by $\frac{1}{3}\binom{n}{2}$, which will be dominated by $\frac{1}{2}\binom{n}{2} - cn^{\frac{3}{2}}$ as n gets large. In Theorem 2.1 we give explicit constructions of tournaments where the size of a minimum feedback arc set is strictly greater than the size of the largest collection of arc disjoint cycles. Furthermore, this construction will produce tournaments that are much smaller than the examples known to exist in [3].

It was shown in [2] that for any acyclic digraph D, there exists a tournament T that has D as a minimum feedback arc set, and hence the reversing number of a digraph is well defined. Furthermore, they showed that $r(D) \leq |A(D)|$. For a general acyclic digraph D, the computation of $r(D)$ has proven to be very difficult, but is tractable for many classes of digraphs. Reversing numbers for many classes of digraphs were determined in [2]. The reversing number for tournaments were investigated in [4] and [1]. The reversing number for a disjoint union of directed stars was determined [6] and the reversing number for powers of a directed Hamiltonian path was studied in [5]. We next note a result from [2] for the reversing number of a path on n vertices. We restate this as our Lemma 1.2 and give a short proof.

Lemma 1.2. $r(P_n) = n - 1$.

Proof. Let $D = P_n$ have vertex set $\{v_1, v_2, \ldots, v_n\}$ and arc set $\{(v_{i+1}, v_i) \mid i = 1, 2, \ldots, n-1\}$. First we will show $r(P_n) \geq n - 1$. Assume $r(P_n) < n - 1$. Then there is some tournament $T(\sigma, D)$ on less than $2n - 1$ vertices that realizes D.

Then by the pigeonhole principle, σ must contain two vertices $\{v_j, v_{j+1}\}$ of P_n such that $\sigma(v_{j+1}) - \sigma(v_j) = 1$. So $D \backslash (v_{j+1}, v_j)$ is a feedback arc set of $T(\sigma, D)$. Hence D is not a minimum feedback arc set of $T(\sigma, D)$.

Next, we will show $r(P_n) \leq n-1$. Let $V(T(\sigma, D)) = V(D)$ along with an additional set of vertices $\{x_1, x_2, \ldots, x_{n-1}\}$ where $\sigma = \langle v_1, x_1, v_2, x_2, \ldots, x_{n-1}, v_n \rangle$. Then $T(\sigma, D)$ contains the following collection of $n-1$ arc disjoint cycles: $\{(v_i, x_i, v_{i+1}) \mid i = 1, 2, \ldots, n-1\}$. Hence any feedback arc set must contain at least $n-1$ arcs and so D is a minimum feedback arc set of $T(\sigma, D)$. This implies $r(P_n) \leq n-1$ and equality follows. ■

We will continue with some elementary results that will be used frequently throughout the paper. We restate Lemma 1 from [2] as our Lemma 1.3.

Lemma 1.3. *Let D' and D be acyclic digraphs with $|V(D')| = |V(D)|$ and $A(D') \subseteq A(D)$. Then $r(D') \leq r(D)$.*

Proof. Let $D' \subseteq D$ be acyclic digraphs on n vertices and let T be a realizing tournament for D. Then consider the tournament T' where $V(T') = V(T)$ and $A(T') = A(T) \backslash [A(D) \backslash A(D')] \cup [A(D) \backslash A(D')]^R$. We will show D' is a minimum feedback arc set of T'. We note that D is a feedback arc set of T and $A(T') \backslash A(D') \cup (A(D'))^R = A(T) \backslash A(D) \cup (A(D))^R$. Assume there is some set F that forms a smaller feedback arc set of T' than D'. Then $F \cup (D \backslash D')$ is a smaller feedback arc set of T than D, a contradiction. ■

The following lemma follows directly from the definitions of reversing number and minimum feedback arc sets.

Lemma 1.4. $r(D + cP_1) = 0 \Leftrightarrow r(D) \leq c$.

Proof. $r(D + cP_1) = 0 \Leftrightarrow D + cP_1$ is a minimum feedback arc set of a tournament T on $|V(D)| + c$ vertices $\Leftrightarrow r(D) \leq c$. ■

In the next lemma, $D_{|X}$ will denote the subdigraph of D induced by the vertices of $X \subseteq V(D)$, and for an ordering of $V(D)$, a consecutive segment of vertices $v_i, v_{i+1}, \ldots, v_{i+j}$ will be denoted by $\langle v_i, v_{i+1}, \ldots, v_{i+j} \rangle$. We restate a result found in [9] as our Lemma 1.5. This lemma is also found in [2].

Lemma 1.5. *If T is a tournament and D is a minimum feedback arc set such that $\pi(v_1) < \pi(v_2) < \cdots < \pi(v_n)$ is the acyclic ordering after reversal of the arcs in D, then for any segment $\langle v_i, v_{i+1}, \ldots, v_{i+j} \rangle = F, D_{|F}$ is a minimum feedback arc set of $T_{|F}$.*

Proof. If an ordering σ is optimal then for any consecutive segment $F, \sigma_{|F}$ must also be optimal. This follows from the fact that if this $\sigma_{|F}$ was not optimal we could create a strictly better ordering $\sigma^{\#}_{|F}$ for the segment F and define σ^* such that $\sigma^*_{|F} = \sigma^{\#}_{|F}$ and $\sigma^*_{|V(T) \backslash F} = \sigma_{|V(T) \backslash F}$. This implies $|INC(\sigma^*, T(\sigma, D))| < |INC(\sigma, T(\sigma, D))|$ which contradicts the optimality of σ. ■

2. DISJOINT UNION OF DIRECTED PATHS

In this section we will classify all tournaments that have a disjoint union of directed paths as a minimum feedback arc set. As a consequence we will precisely determine the reversing number for all digraphs consisting of a disjoint union of directed paths.

Let $D = \sum_{i=1}^{m} P_{k_i}$, where $k_1 \geq k_2 \geq \cdots \geq k_m$. Let the vertices of each path P_{k_i} be $v_{i,1}, v_{i,2}, \ldots, v_{i,k_i}$ and let the arcs of each path P_{k_i} be $(v_{i,j+1}, v_{i,j})$ for all $j = 1, 2, \ldots, k_i - 1$.

Example 2.1. Let $D = P_2 + P_2 + P_2$. Then $r(D) = 0$.

Let T be the tournament shown in Figure 2. Then D is a feedback arc set of T since the reversal of the arcs of D creates the tournament where all of the arcs are directed from left to right, which is clearly acyclic. There are 3 arc disjoint cycles $(v_{1,1}, v_{3,1}, v_{1,2})$, $(v_{2,1}, v_{1,2}, v_{2,2})$, and $(v_{3,1}, v_{2,2}, v_{3,2})$, so any feedback arc set of T must contain at least three arcs. Hence D is a minimum feedback arc set of T. Then $|V(D)| = |V(T)|$ and $r(D) = 0$.

As we noted in Lemma 1.1 the size of a minimum feedback arc set in a realizing tournament T is greater than or equal to the size of a largest collection of arc disjoint cycles in T. This next theorem will give explicit examples where we have strict inequality. Given a tournament $T(\sigma, D)$ we denote the size of a largest collection of arc disjoint cycles in this tournament by $|C_{\max}(T(\sigma, D))|$. The next theorem will present tournaments with a minimum feedback arc set of size $4k - 2$ but where the size of the largest collection of arc disjoint cycles is $3k - 1 + \lfloor \frac{k-1}{2} \rfloor$ for all integers $k \geq 2$. We will establish the upper bound on the cycles here and the lower bound on the size of the minimum feedback arc set will follow later in the proof of Lemma 3.3. We note that none of the later lemmas and theorems will use Theorem 2.1. We state it here simply to give some justification for the need of methods other than seeking a collection of $|A(D)|$ arc disjoint cycles in a tournament realizing D.

Theorem 2.1. *Let $D = P_{2k+1} + 2P_k$ where $k \geq 2$. Let σ be an ordering for $V(D)$ where $\sigma(v_{1,i}) = 2i - 1$ for $i = 1, 2, \ldots, 2k + 1$, $\sigma(v_{2,i}) = 4i - 2$ for $i = 1, 2, \ldots, k$, and $\sigma(v_{3,i}) = 4i$ for $i = 1, 2, \ldots, k$. Then D is a minimum feedback arc set of $T(\sigma, D)$ and $|C_{\max}(T(\sigma, D))| = 3k - 1 + \lfloor \frac{k-1}{2} \rfloor < 3k - 2 = |A(D)|$.*

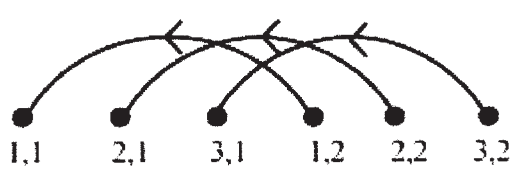

FIGURE 2. $T(\sigma, D)$; all arcs that are not drawn are directed from left to right and $v_{i,j}$ is denoted as i,j.

Proof. We will first show that the largest collection of arc disjoint cycles in $T(\sigma, D)$ has size $3k-1+\lfloor\frac{k-1}{2}\rfloor$. The proof will be completed in Lemma 3.3 when we establish that D is, in fact, a minimum feedback arc set of $T(\sigma, D)$. At times it may be useful to refer to Figure 3.

We seek to show $|C_{\max}(T(\sigma, D))| \geq 3k-1+\lfloor\frac{k-1}{2}\rfloor$. We note $T(\sigma, D)$ contains the following set of arc disjoint cycles: $\{(v_{1,i}, v_{2,\frac{i+1}{2}}, v_{1,i+1}) \mid i = 1, 3, \ldots, 2k-1\} \cup \{(v_{1,i}, v_{3,\frac{i}{2}} v_{1,i+1}) | i = 2, 4, \ldots, 2k\} \cup \{(v_{2,j}, v_{3,j}, v_{2,j+1}) \mid j = 1, 2, \ldots, k-1\} \cup \{(v_{3,2j-1}, v_{1,4j}, v_{3,2j+1}, v_{3,2j}) \mid j = 1, 2, \ldots, \lfloor\frac{k-1}{2}\rfloor\}$. Hence the lower bound follows.

Next, we will show $|C_{\max}(T(\sigma, D))| \leq 3k-1+\lfloor\frac{k-1}{2}\rfloor$. The proof will be by induction on k, using k to get to $k+2$, so we present two base cases. We begin with the base case where $D = P_5 + P_2 + P_2$. Since each cycle must contain at least one arc from $A(D)$, $|C_{\max}(T(\sigma, D))| \leq 6$. If we hope to obtain a collection of six arc disjoint cycles, then each cycle must contain exactly one arc from $A(D)$. The arcs $(v_{1,2}, v_{1,1}), (v_{1,3}, v_{1,2}), (v_{1,4}, v_{1,3})$, and $(v_{1,5}, v_{1,4})$ must be contained in the cycles $(v_{1,1}, v_{2,1}, v_{1,2}), (v_{1,2}, v_{3,1}, v_{1,3}), (v_{1,3}, v_{2,2}, v_{1,4})$, and $(v_{1,4}, v_{3,2}, v_{1,5})$, respectively. Then if both of the arcs $(v_{2,2}, v_{2,1})$ and $(v_{3,2}, v_{3,1})$ are in cycles that are arc disjoint from the four cycles mentioned above, then both $(v_{2,2}, v_{2,1})$ and $(v_{3,2}, v_{3,1})$ must be in cycles that contain the arc $(v_{3,1}, v_{2,2})$. Hence $|C_{\max}(T(\sigma, D))| \leq 5$.

The second base case is where $D = P_7 + P_3 + P_3$. Since each cycle must contain at least one arc from $A(D)$, $|C_{\max}(T(\sigma, D))| \leq 10$. If we hope to obtain a collection of 10 arc disjoint cycles, then each cycle must contain exactly one arc from $A(D)$. The arcs $(v_{1,2}, v_{1,1}), (v_{1,3}, v_{1,2}), (v_{1,4}, v_{1,3}), (v_{1,5}, v_{1,4}), (v_{1,6}, v_{1,5})$, and $(v_{1,7}, v_{1,6})$ must be contained in the cycles $(v_{1,1}, v_{2,1}, v_{1,2}), (v_{1,2}, v_{3,1}, v_{1,3}), (v_{1,3}, v_{2,2}, v_{1,4}), (v_{1,4}, v_{3,2}, v_{1,5}), (v_{1,5}, v_{2,3}, v_{1,6})$, and $(v_{1,6}, v_{3,3}, v_{1,7})$, respectively. If both of the arcs $(v_{2,2}, v_{2,1})$ and $(v_{3,2}, v_{3,1})$ are in cycles that are arc disjoint from the six cycles mentioned above, then both $(v_{2,2}, v_{2,1})$ and $(v_{3,2}, v_{3,1})$ must be in cycles that contain the arc $(v_{3,1}, v_{2,2})$. Hence $|C_{\max}(T(\sigma, D))| \leq 9$.

Let D' be the digraph where $V(D') = V(D) \cup \{v_{2,k+1}, v_{1,2k+2}, v_{3,k+1}, v_{1,2k+3}, v_{2,k+2}, v_{1,2k+4}, v_{3,k+2}, v_{1,2k+5}\}$, and $A(D') = A(D) \cup \{(v_{2,k+1}, v_{2,k}), (v_{1,2k+2}, v_{1,2k+1}), (v_{1,2k+3}, v_{1,2k+2}), (v_{2,k+2}, v_{2,k+1}), (v_{3,k+2}, v_{3,k+1}), (v_{1,2k+5}, v_{1,2k+4})\}$ and let σ'

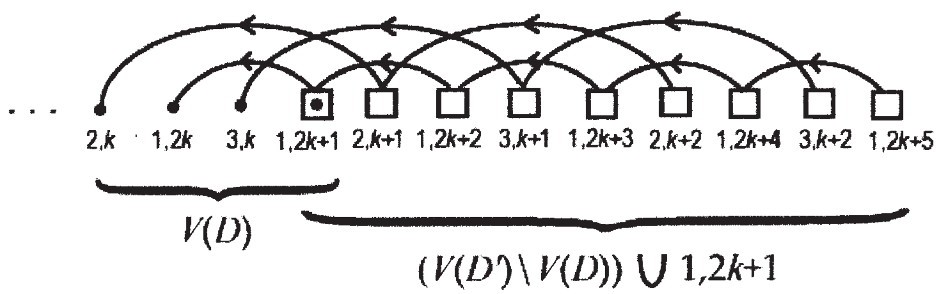

FIGURE 3. Note all arcs that are not drawn are directed from left to right and $v_{i,j}$ is denoted as i, j.

be the ordering that is identical to σ on $V(D)$ and $\sigma^{-1}(|V(D)|+1) = v_{2,k+1}$, $\sigma^{-1}(|V(D)|+2) = v_{1,2k+2}, \sigma^{-1}(|V(D)|+3) = v_{3,k+1}, \sigma^{-1}(|V(D)|+4) = v_{1,2k+3}$, $\sigma^{-1}(|V(D)|+5) = v_{2,k+2}, \sigma^{-1}(|V(D)|+6) = v_{1,2k+4}, \sigma^{-1}(|V(D)|+7) = v_{3,k+2}$, and $\sigma^{-1}(|V(D)|+8) = v_{1,2k+5}$. To show that the hypothesis can be extended from k to $k+2$ we establish that the size of the largest collection of arc disjoint cycles in $T(\sigma', D')$ is $3(k+2)-1+\lfloor\frac{(k+2)-1}{2}\rfloor$.

In any collection of arc disjoint cycles there are at most three types of cycles. Let $V_1 = V(D)$ and let $V_2 = (V(D')\backslash V(D)) \cup v_{1,2k+1}$. A cycle can have all of its vertices in V_1, all of its vertices in V_2 or at least one vertex in each of the two sets. By the induction hypothesis the largest number of arc disjoint cycles that can have all of their vertices in V_1 is $3k-1+\lfloor\frac{k-1}{2}\rfloor$. By the first base case the subtournament induced by V_2 contains at most 5 arc disjoint cycles. Note that there are only two arcs of D that go between the two sets. Hence there are at most two cycles with vertices in each of the sets V_1 and V_2 that can be added to our collection. Hence the largest collection of arc disjoint cycles in $T(\sigma', D')$ has a size less than or equal to $3k-1+\lfloor\frac{k-1}{2}\rfloor+7 = 3(k+2)-1+\lfloor\frac{(k+2)-1}{2}\rfloor$. ■

A. An Extension Lemma

We will consider some methods to show certain digraphs have reversing number zero even though the largest set of arc disjoint cycles in a tournament realizing D has cardinality strictly less than $|A(D)|$. Let D be a disjoint union of directed paths with an ordering σ such that D is a minimum feedback arc set of $T(\sigma, D)$. We will extend D to a superdigraph D' by adding a vertex w, and extend σ to another ordering σ' on $V(D')$ where $\sigma'_{|V(D)} = \sigma_{|V(D)}$, and $\sigma'(w) = |V(T(\sigma, D))|+1$. In Lemma 2.2 we will present sufficient conditions that will allow us to extend σ to σ' and D to D' so that D' is a minimum feedback arc set of $T(\sigma', D')$.

Recall that $|INC(\sigma', T(\sigma', D'))| = |A(D')|$. If it can be shown for any ordering π of $V(D')$ that $|INC(\pi, T(\sigma', D'))| \geq |INC(\sigma', T(\sigma', D'))|$ it will follow that D' is a minimum feedback arc set of $T(\sigma', D')$. The following lemma shows that unless π has a certain form we immediately have $|INC(\pi, T(\sigma', D'))| \geq |INC(\sigma', T(\sigma', D'))|$.

Lemma 2.1. *Let D be a disjoint union of directed paths that is a minimum feedback arc set of $T(\sigma, D)$. Let $|V(D)| = n$ and $\sigma(v_{r,k_r}) \neq n$. Then let D' be the digraph where $V(D') = V(D) \cup v_{r,k_r+1}$ and $A(D') = A(D) \cup (v_{r,k_r+1}, v_{r,k_r})$. Let σ' be the ordering such that $\sigma'(v_{r,k_r+1}) = n+1$ and $\sigma'(v) = \sigma(v)$ for all other $v \in V(D)$. Let π be an optimal ordering for $V(D')$ such that $|INC(\pi, T(\sigma', D'))| < |INC(\sigma', T(\sigma', D'))|$. Then $\pi(v_{r,k_r+1}) = n$ and $\pi(v_{r,k_r}) = n+1$.*

Proof. We will show that if either $\pi(v_{r,k_r+1}) \neq n$ or $\pi(v_{r,k_r}) \neq n+1$, then $|INC(\pi, T(\sigma', D'))| \geq |A(D')|$. We will start by considering the case where $\pi(v_{r,k_r+1}) \neq n$. If $\pi(v_{r,k_r+1}) = n+1$, then there are at least $|INC(\sigma, T(\sigma, D))|$ inconsistencies from $V(D)$, plus one additional inconsistency since $\pi(v_{r,k_r+1}) > \pi(v_{r,k_r})$, and hence $|INC(\pi, T(\sigma', D'))| \geq |A(D')|$. If $\pi(v_{r,k_r+1}) < n$ then there are at least $|INC(\sigma, T(\sigma, D))|$ inconsistencies from $V(D)$, plus one additional inconsistency since there is some vertex $v \neq v_{r,k_r}$, such that $\pi(v) > \pi(v_{r,k_r+1})$. This gives a total of $|A(D')|$ inconsistencies. Hence we will proceed with $\pi(v_{r,k_r+1}) = n$. If $\pi(v_{r,k_r}) < n$ then we will have $|INC(\sigma, T(\sigma, D))|$ inconsistencies from $V(D)$, plus an additional inconsistency since $\pi(v_{r,k_r+1}) > \pi(v_{r,k_r})$. This gives a total of $|A(D')|$ inconsistencies. ■

Given that $D = \sum_{i=1}^{m} P_{k_i}$ is a minimum feedback arc set of $T(\sigma, D)$, our goal is to extend σ to σ' and D to a superdigraph D' such that D' is a minimum feedback arc set of $T(\sigma', D')$. We will extend D to a superdigraph D' by adding a vertex w and possibly an arc from w to a member of $V(D)$. We will extend σ to another ordering σ' on $V(D')$ where $\sigma'_{|V(D)} = \sigma_{|V(D)}$, and $\sigma'(w) = |V(T(\sigma, D))| + 1$. By definition D' is a feedback arc set of $T(\sigma', D')$.

We first assume that D is a disjoint union of directed paths that is a minimum feedback arc set of $T(\sigma, D)$. If neither of the last two vertices in σ are on the path P_{k_i} then we will add one vertex to the beginning of the path P_{k_i}. We note that this new vertex when considering σ' is the right most of any other vertex in P_{k_i}. The next lemma will show that we can then extend σ to σ' and D to D' so that D' is a minimum feedback arc set of $T(\sigma', D')$.

Lemma 2.2 (Extension Lemma). *Let D be a disjoint union of directed paths that is a minimum feedback arc set of $T(\sigma, D)$ and let $n = |V(T(\sigma, D))| \geq 3$. Let $v_{r,k_{r+1}}$ be a vertex of D that is not equal to either $\sigma^{-1}(n)$ or $\sigma^{-1}(n-1)$. Let D' be the digraph where $V(D') = V(D) \cup v_{r,k_r+1}$ and $A(D') = A(D) \cup (v_{r,k_r+1}, v_{r,k_r})$. Let σ' be the ordering such that $\sigma'(v_{r,k_r+1}) = n + 1$ and $\sigma'(v) = \sigma(v)$ for all other $v \in V(D)$. Then D' is a minimum feedback arc set of $T(\sigma', D')$.*

Proof. We will show for any ordering π, $|INC(\pi, T(\sigma', D'))| \geq |INC\ (\sigma', T\ (\sigma', D'))|$. Since $\sigma(v_1) < n - 1$ we have the following general structure for $T(\sigma, D)$: $\langle X, v_{r,k_r}, Y \rangle$, where $|Y| \geq 2$. Let v_{a,k_l} and v_{b,k_j} be the first and second vertices of Y, respectively, that appear in the ordering σ. Let $D_{|X}$ and $D_{|Y}$ be the induced subdigraphs of D on the vertices of X and Y, respectively. We will first consider the case where k_r and k_j are at least 2. The other cases will be handled later.

We note that $|A(D')| \leq |A(D_{|X})| + |Y| + 2$ since there are $|A(D_{|X})|$ arcs with tails in X and at most $|Y| + 2$ arcs with tails in $Y \cup \{v_{r,k_r+1}, v_{r,k_r}\}$. By Lemma 2.1 we may assume $\pi(v_1) = n + 1$ and $\pi(v_2) = n$. We next consider $|INC(\pi, T\ (\sigma', D'))|$. Then there are $|A(D_{|X})|$ inconsistencies from arcs with both their head and tail in X. Since $\pi(v_{r,k_r}) > \pi(v_y)$ for all vertices $v_y \in Y$ and $\pi(v_{r,k_r}) > \pi(v_{r,k_r-1})$, there are $|Y| + 1$ additional inconsistencies from arcs that have v_{r,k_r} at one end. Finally, there is one more inconsistent arc from the triangle $(v_{b,k_j-1}, v_{a,k_i}, v_{b,k_j})$, which has either one end in X and the other end in Y, or both ends in Y. To see that we have not double counted any of the above inconsistencies, we note that there are $|A(D_{|X})|$ inconsistencies that only involve vertices from X, $|Y|$ inconsistencies that involve v_{r,k_r} and a vertex from

Y, the inconsistency $\pi(v_{r,k_r}) > \pi(v_{r,k_r-1})$ and one inconsistency from the triangle $(v_{b,k_j-1}, v_{a,k_i}, v_{b,k_j})$. Hence $|INC(\pi, T(\sigma', D'))| \geq |A(D_{|X})| + |Y| + 2$.

If $k_r = 1$ and $k_j \geq 2$ or $k_r \geq 2$ and $k_j = 1$, then $A(D')$ contains one less arc than in the above general case and we see $|INC(\pi, T(\sigma', D'))|$ and $|A(D_{|X})| + |Y| + 2$ each decrease by 1. If $(v_{r,k_r}, v_{r,k_r-1}) \notin A(D)$ and $(v_{b,k_j}, v_{b,k_j-1}) \notin A(D)$, then $A(D')$ contains two less arcs than in the above general case and $|INC(\pi, T(\sigma', D'))|$ and $|A(D_{|X})| + |Y| + 2$ are each decreased by 2. ∎

B. Forbidden Strings

Let D be a disjoint union of directed paths and let D be a feedback arc set of $T(\sigma, D)$. In Lemma 1.5 we noted that if D is a minimum feedback arc set of $T(\sigma, D)$, then for any segment of vertices F appearing in σ, $|INC(\pi_{|F}, T(\sigma_{|F}, D_{|F}))| \geq |INC(\sigma_{|F}, T(\sigma_{|F}, D_{|F}))|$ for any ordering π of $V(D)$. Hence if a segment of vertices, S, appears in σ and there exists an ordering π' such that $|INC(\pi'_{|S}, T(\sigma_{|S}, D_{|S}))| < |INC(\sigma_{|S}, T(\sigma_{|S}, D_{|S}))|$, then D is not a minimum feedback arc set of $T(\sigma, D)$. In this section we will focus on these "non-optimal" segments.

Given an ordering σ consistent with D, we will define a *forbidden string* $S(\sigma, D)$ to be a labeling that corresponds to a segment S of vertices in σ where there exists an ordering π such that $|INC(\pi_{|S}, T(\sigma_{|S}, D_{|S}))| < |INC(\sigma_{|S}, T(\sigma_{|S}, D_{|S}))|$. This string is called forbidden because if D is a minimum feedback arc set of $T(\sigma, D)$, then $S(\sigma, D)$ cannot contain such a string. We will define two types of forbidden strings and use them to describe all tournaments that have a disjoint union of directed paths as a minimum feedback arc set.

We will formally define these forbidden strings by considering the segment of vertices in the ordering σ. Given a digraph D consisting of a disjoint union of directed paths and an ordering σ, we can define a string $S(\sigma, D) = \langle s_1, s_2, \ldots, s_n \rangle$ where $s_j = i$ if and only if $\sigma^{-1}(j)$ is on the directed path P_{k_i}. We note that this correspondence is reversible, that is, given a string $S_n = \langle s_1, s_2, \ldots, s_n \rangle$ we can appropriately define the unique digraph D consisting of a disjoint union of directed paths and a unique ordering σ. The ordering σ can constructed by replacing the jth i with $v_{i,j}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i$. Then $A(D)$ consists of the arcs $\{(v_{i,j+1}, v_{i,j}) | j = 1, 2, \ldots, k_i - 1\}$. From this we can define the tournament $T(\sigma, D)$ that has D as a feedback arc set.

We show the correlation between D, σ and $S(\sigma, D)$ in the following example.

Example 2.2. $S_n = \langle 1, 2, 1, 3, 2, 1, 2, 3, 1, 2, 1 \rangle$. Then $D = P_5 + P_4 + P_2$ and $\sigma = \langle v_{1,1}, v_{2,1}, v_{1,2}, v_{3,1}, v_{2,2}, v_{1,3}, v_{2,3}, v_{3,2}, v_{1,4}, v_{2,4}, v_{1,5} \rangle$. $T(\sigma, D)$ is the tournament presented in Figure 4. Again we note that all arcs not drawn are directed from left to right.

Definition 2.1. *We define a forbidden string of type I to be a string of vertices* $\langle i, i \rangle$ *occurring in* S_n*. This corresponds to two vertices from the same feedback path, that appear consecutively in* σ*.*

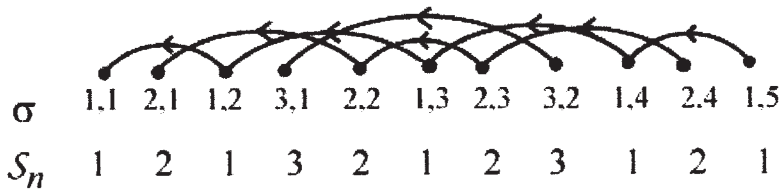

FIGURE 4. Correspondence between σ, S_n, and $T(\sigma, D)$.

We show a corresponding subdigraph of a forbidden string of type I in Figure 5. This idea of this forbidden string was alluded to in the proof of Lemma 1.2, but it is helpful to think of this idea in the context of player rankings. A forbidden string of type I is then a ranking of players where two players are ranked consecutively and the lower ranked player beats the higher ranked player. Clearly this ranking is not optimal for the ranking method we are using, because we could switch these two players in the ranking and keep the ranks of all other players fixed and obtain a new ranking with strictly fewer inconsistencies. We state this property in the context of minimum feedback arc sets below.

Lemma 2.3. *Let D be a disjoint union of directed paths and let σ be an ordering consistent with D. If $S(\sigma, D)$ contains a forbidden string of type I, then D is not a minimum feedback arc set of $T(\sigma, D)$.*

Proof. Assume that $S(\sigma, D)$ contains the string $\langle i, i \rangle$. Then σ must contain two consecutive vertices that belong to the same path in D. An ordering π, which interchanges the order of these two vertices in σ and maintains the same ordering as σ for all other vertices implies $|INC(\pi, T(\sigma, D))| = |INC(\sigma, T(\sigma, D))| - 1$. Hence D is not a minimum feedback arc set of $T(\sigma, D)$. ∎

Definition 2.2. *We define a forbidden string type II to be a string $S_k = \langle s_1, s_2, \ldots, s_k \rangle$ where $k \geq 4$, that does not contain a forbidden string of type I and where $s_3 = s_1, s_k = s_{k-2}$ and $s_{k-2i+1} = s_{k-2i-2}$ for $i = 1, 2, \ldots, \frac{k}{2} - 2$.*

For the corresponding subdigraphs, see Figure 6. We note that these are not necessarily induced subdigraphs since the subdigraph may contain additional arcs that join the different directed paths (without creating a subdigraph corresponding to a forbidden string of type I). These subdigraphs with additional arcs will still contain a subdigraph consisting a disjoint union of directed paths that correspond to a forbidden string of Type II.

FIGURE 5. Subdigraph corresponding to a forbidden string of Type I with respect to the ordering σ.

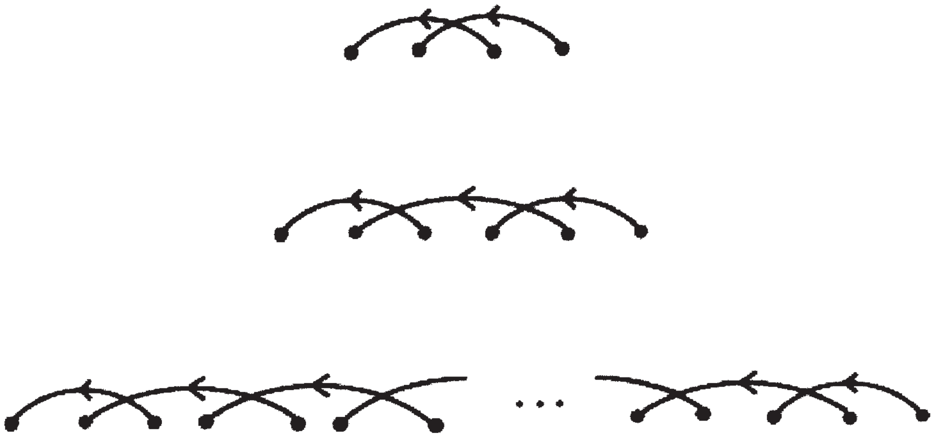

FIGURE 6. Subdigraphs corresponding to forbidden strings of type II with respect to the ordering σ.

In Lemma 2.4 we will show that if D is a disjoint union of directed paths, σ is an ordering consistent with D and $S(\sigma, D)$ contains a forbidden string of type II, then D is not a minimum feedback arc set of $T(\sigma, D)$. However we will precede this lemma with an example illustrating the main ideas of Lemma 2.4 and its proof.

Example 2.3. Let $D = P_4 + P_2 + P_2$ and let $\sigma = \langle v_{1,1}, v_{2,1}, v_{1,2}, v_{3,1}, v_{2,2}, v_{1,3}, v_{3,2}, v_{1,4}\rangle$ be consistent with D. Then $S(\sigma, D) = \langle 1, 2, 1, 3, 2, 1, 3, 1\rangle$. See Figure 7

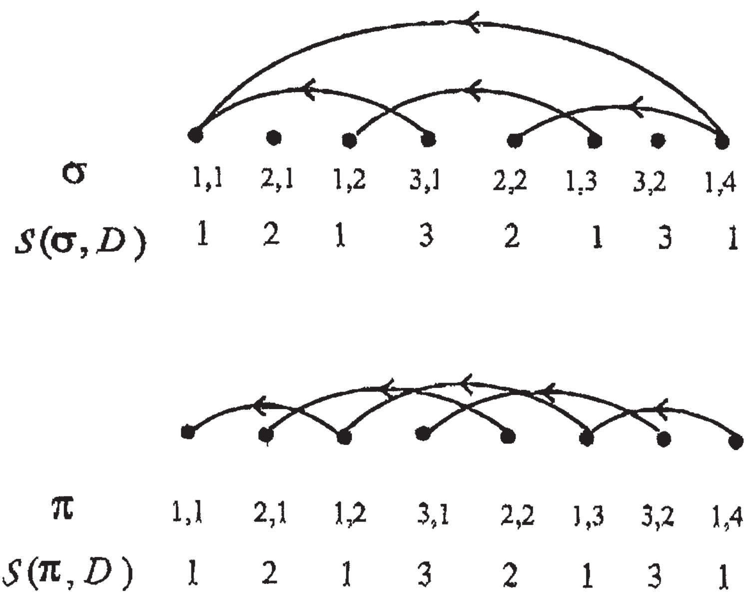

FIGURE 7. $D = P_4 + P_2 + P_2$ with two orderings σ and π.

and note all arcs that are not drawn are directed from left to right. We will show D is not a minimum feedback arc set of $T(\sigma, D)$. We consider the ordering $\pi = \langle v_{1,2}, v_{1,1}, v_{2,2}, v_{2,1}, v_{3,2}, v_{3,1}, v_{1,4}, v_{1,3} \rangle$. We compare $INC(\pi, T(\sigma, D))$ and $INC(\sigma, T(\sigma, D))$. Let $A = [INC(\pi, T(\sigma, D))] \backslash [INC(\sigma, T(\sigma, D))]$ and $B = [INC(\sigma, T(\sigma, D))] \backslash [INC(\pi, T(\sigma, D))]$.

Note that the set A consists of the arcs directed from right to left in $T(\pi, D)$ and left to right in $T(\sigma, D)$. These arcs are $(v_{1,3}, v_{3,2}), (v_{3,1}, v_{2,2})$, and $(v_{2,1}, v_{1,2})$. The set B consists of the arcs that are directed from right to left in $T(\sigma, D)$ and left to right in $T(\pi, D)$. These arcs are $(v_{1,2}, v_{1,1}), (v_{1,4}, v_{1,3}), (v_{2,2}, v_{2,1})$, and $(v_{3,2}, v_{3,1})$. Since $|B| - |A| = 4 - 3 = 1$, we have that $|INC(\pi, T(\sigma, D))| = |INC(\sigma, T(\sigma, D))| - 1$, and hence D is not a minimum feedback arc set of the tournament $T(\sigma, D)$.

We note that this does not imply that $r(P_4 + P_2 + P_2) > 0$, only that this tournament does not realize $D = P_4 + P_2 + P_2$. In fact, $r(P_4 + P_2 + P_2) = 0$ which will follow from Theorem 3.1.

Lemma 2.4. *Let D be a disjoint union of directed paths and let σ be an ordering consistent with D. If $S(\sigma, D)$ contains a forbidden string of type II, then D is not a minimum feedback arc set of $T(\sigma, D)$.*

Proof. Assume that $S(\sigma, D)$ contains a forbidden string of type II, $S_k = \langle s_1, s_2, \ldots, s_k \rangle$. Let $\langle s_1, s_2, \ldots, s_k \rangle$ correspond to vertices $\sigma^{-1}(i), \sigma^{-1}(i+1), \ldots, \sigma^{-1}(i+k-1)$ in the ordering σ. To keep the notation simple, we will consider the case where $i = 1$ and note that the other cases follow from Lemma 1.5. We consider the ordering π, where $\pi^{-1}(k) = \sigma^{-1}(k-2), \pi^{-1}(k-1) = \sigma^{-1}(k)$, $\pi^{-1}(k-2j) = \sigma^{-1}(k-2j-2)$, and $\pi^{-1}(k-2j-1) = \sigma^{-1}(k-2j+1)$, for $j = 1, 2, \ldots, \frac{k}{2} - 2$ and $\pi^{-1}(2) = \sigma^{-1}(1), \pi^{-1}(1) = \sigma^{-1}(3)$. We then compare $INC(\pi, T(\sigma, D))$ and $INC(\sigma, T(\sigma, D))$. Let $A = [INC(\pi, T(\sigma, D))] \backslash [INC(\sigma, T(\sigma, D))]$ and $B = [INC(\sigma, T(\sigma, D))] \backslash [INC(\pi, T(\sigma, D))]$. Then we consider the set A, which consists of arcs that are directed from right to left in $T(\pi, D)$ and left to right in $T(\sigma, D)$. These arcs are $\{(\pi^{-1}(k-2j), \pi^{-1}(k-2j-3))$ for $j = 0, 1, \ldots, \frac{k}{2} - 2\}$. The set B consists of the arcs that are directed from right to left in $T(\sigma, D)$ and left to right in $T(\pi, D)$. These arcs are $\{(\pi^{-1}(k-2j), \pi^{-1}(k-2j-1))$ for $j = 0, 1, \ldots, \frac{k}{2} - 1\}$. Since $|B| - |A| = 1$, we have that $|INC(\pi, T(\sigma, D))| = |INC(\sigma, T(\sigma, D))| - 1$, and hence D is not a minimum feedback arc set of $T(\sigma, D)$. ∎

We next observe a trivial extension. Let D be a minimum feedback arc set of $T(\sigma, D)$. Let D' be the digraph where $V(D') = V(D) \cup w$, for a vertex $w \notin V(D)$ and $A(D') = A(D)$. Define σ' to be the ordering such that $\sigma'(w) = |V(D)| + 1$ and $\sigma'(v) = \sigma(v)$ for all other vertices $v \in V(D)$. Then we can immediately conclude that D' is a minimum feedback arc set of $T(\sigma', D')$.

Theorem 2.2 will describe all tournaments that have a digraph D consisting of disjoint union of directed paths as a minimum feedback arc set. We will prove that they are exactly the tournaments $T(\sigma, D)$ where σ is the acyclic ordering obtained upon reversal of the arcs of D and $S(\sigma, D)$ does not contain either type of forbidden string, that is, all tournaments that do not contain a subdigraph (not necessarily induced) shown in Figure 5 or 6.

Theorem 2.2. *Let D be a disjoint union of directed paths and let σ be an ordering consistent with D. Then D is a minimum feedback arc set of $T(\sigma, D)$ if and only if the sequence $S(\sigma, D)$ does not contain a forbidden string of type I or II.*

Proof. ($\Rightarrow$) This direction follows from Lemmas 2.3 and 2.4. ($\Leftarrow$) To prove the other direction, we will proceed by induction on $n = |V(D)|$. We will start with a few base cases. If $|V(D)| \leq 2$ and $S(\sigma, D)$ does not contain a forbidden string, then $|A(D)| = 0$, and our result is trivial.

Assume the hypothesis is true for all digraphs with n vertices and we will show that it holds for all digraphs with $n+1$ vertices. Let $S(\sigma, D) = \langle s_1, s_2, \ldots, s_{n+1} \rangle$ be a sequence that does not contain a forbidden string of type I or II. If $\sigma^{-1}(n+1)$ is a vertex that is not on the same path as a vertex in $V(D)$, then by a trivial extension D is a minimum feedback arc set of $T(\sigma, D)$. Let $S(\sigma^{\#}, D^{\#}) = \langle s_1, s_2, \ldots, s_n \rangle$. Note that $S(\sigma^{\#}, D^{\#})$ does not contain a forbidden string and by induction $D^{\#}$ is a minimum feedback arc set of $T(\sigma^{\#}, D^{\#})$. Since $S(\sigma, D)$ does not contain a forbidden string of type I, $s_{n+1} \neq s_n$, and $s_n \neq s_{n-1}$. Consider the case where $s_{n+1} \neq s_{n-1}$. Since $D^{\#}$ is a minimum feedback arc set of $T(\sigma^{\#}, D^{\#})$ and s_{n+1} is not equal to either s_n or s_{n-1}, we can apply the Lemma 2.2 to conclude that D is a minimum feedback arc set of $T(\sigma, D)$.

Hence we will proceed with the case where $s_{n+1} = s_{n-1}$. If $s_n = s_{n-2}$, then $S(\sigma, D)$ would contain $\langle s_{n-2}, s_{n-1}, s_n, s_{n+1} \rangle$, where $s_{n+1} = s_{n-1}$ and $s_n = s_{n-2}$, which is a forbidden string of type II. Hence we will proceed with $s_n \neq s_{n-2}$. Let $S(\sigma^*, D^*) = (s_1, s_2, \ldots, s_{n-2}, s_n)$. Since $s_n \neq s_{n-2}, S(\sigma^*, D^*)$ cannot contain a forbidden string of type I. We will show next that $S(\sigma^*, D^*)$ cannot contain a forbidden string of type II. If $S(\sigma^*, D^*)$ contained a forbidden string of type II, then $s_n = s_{n-3}, s_{n-2i} = s_{n-2i-3}$ for $i = 1, 2, \ldots, \lfloor \frac{n-k-3}{2} \rfloor$ and $s_k = s_{k-2}$. This implies $s_{n+1} = s_{n-1}$ and $s_{n-2i} = s_{n-2i-3}$ for $i = 0, 1, \ldots, \lfloor \frac{n-k-3}{2} \rfloor$, which is a forbidden string of type II contained in $S(\sigma, D)$. This is a contradiction. Hence D^* is a minimum feedback arc set of $T(\sigma^*, D^*)$.

We seek to show that for any ordering π of $V(D), |INC(\pi, T(\sigma, D))| \geq |INC(\sigma, T(\sigma, D))|$. If $\pi^{-1}(n+1)$ and $\pi^{-1}(n)$ do not equal $\sigma^{-1}(n-1)$ and $\sigma^{-1}(n+1)$, respectively then we are done by Lemma 2.1. We will proceed with $\pi^{-1}(n+1) = \sigma^{-1}(n-1)$ and $\pi^{-1}(n) = \sigma^{-1}(n+1)$. This implies that the vertices of D^* appear consecutively in π.

We will consider two cases. If $s_{n-1} = s_i$ for some $i < n-1$, then $|A(D^*)| = |A(D)| - 2$. Since D^* is a minimum feedback arc set of $T(\sigma^*, D^*)$, and since the vertices of D^* appear consecutively in π, there are at least $|A(D^*)|$ inconsistent arcs with both ends in $V(D^*)$. Then we have two more inconsistencies since $\sigma^{-1}(n-1)$ beats $\sigma^{-1}(n)$ in the ordering π and $\sigma^{-1}(n-1)$ beats $\sigma^{-1}(i)$. Hence $|INC(\pi, T(\sigma, D))| \geq |A(D^*)| + 2 = |A(D)|$ and so D is a minimum feedback arc set of $T(\sigma, D)$.

If $s_{n-1} \neq s_i$ for any $i < n-1$, then $|A(D^*)| = |A(D)| - 1$. Since D^* is a minimum feedback arc set of $T(\sigma^*, D^*)$, and since the vertices of D^* appear consecutively in π, there are at least $|A(D^*)|$ inconsistent arcs with both ends in $V(D^*)$. Then there is an additional inconsistency since $\sigma^{-1}(n-1)$ beats $\sigma^{-1}(n)$ in the ordering π. Then for any ordering π we have that $|INC(\pi, T(\sigma, D))| \geq |A(D)|$, and D is a minimum feedback arc set of $T(\sigma, D)$. ■

3. REVERSING NUMBER OF A DIGRAPH

The reversing number of a digraph D was defined in [2] to be $r(D) = |V(T)| - |V(D)|$, where T is a smallest sized tournament that has D as a minimum feedback arc set. As a consequence of our classification of all tournaments having a disjoint union of directed paths as a minimum feedback arc set, we are able to obtain a new result for the reversing number of a digraph. In this section we will give precise reversing numbers for all digraphs consisting of a disjoint union of directed paths.

We begin by using forbidden strings of types I and II to generate lower bounds for the reversing number of a disjoint union of directed paths.

Lemma 3.1. *Let* $D = \sum_{i=1}^{m} P_{k_i}$, *where* $k_1 \geq k_2 \geq \cdots \geq k_m$. *Then* $r(D) \geq k_1 - 1 - \sum_{i=2}^{m} k_i$.

Proof. Let W be a set of isolated vertices disjoint from $V(D)$. Assume $|W| < k_1 - 1 - \sum_{i=2}^{m} k_i$, that is, where the number of extra vertices is too small. Then let σ be any ordering of $V(D) \cup W$, such that D is a feedback arc set of $T(\sigma, D)$. Then we have less than $k_1 - 1$ vertices in σ that are not on the path P_{k_1}. Then any ordering σ contains at least two more vertices that are on the path P_{k_1} than vertices that are not on the path, there must be two vertices from P_k, that appear consecutively in σ. This implies $S(\sigma, D)$ contains a forbidden string of type I and hence by Lemma 2.3, D is not a minimum feedback arc set of $T(\sigma, D)$. This is a contradiction. Hence $r(D) \geq k_1 - 1 - \sum_{i=2}^{m} k_i$. ■

Lemma 3.2. *Let* $D = \sum_{i=1}^{m} P_{k_i}$, *where* $k_1 \geq k_2 \geq \cdots \geq k_m$. *Then* $r(D) \geq \lfloor \frac{k_1+k_2-1}{3} \rfloor - \sum_{i=3}^{m} k_i$.

Proof. Let W be a set of isolated vertices disjoint from $V(D)$. Let σ be any ordering of $V(D) \cup W$, such that D is a minimum feedback arc set of $T(\sigma, D)$. Assume $|W| < \lfloor \frac{k_1+k_2-1}{3} \rfloor - \sum_{i=3}^{m} k_i$. Since $|W| + \sum_{i=3}^{m} k_i < \lfloor \frac{k_1+k_2-1}{3} \rfloor$, any ordering σ must contain a segment consisting of four vertices from the first two paths. Let W be a set of isolated vertices disjoint from $V(D)$. Then let σ be any ordering of $V(D) \cup W$, such that D is a feedback arc set of $T(\sigma, D)$. Then one or more of the forbidden strings $\langle i, i \rangle$ or $\langle i, j, i, j \rangle$ appears in $S(\sigma, D)$. Thus by Lemmas 2.3 and 2.4, D is not a minimum feedback arc set of $T(\sigma, D)$. This is a contradiction. Hence $r(D) \geq \lfloor \frac{k_1+k_2-1}{3} \rfloor - \sum_{i=3}^{m} k_i$. ■

A. Reversing Number Zero

Our next two lemmas show sufficient conditions for a digraph consisting of a disjoint union of directed paths to have reversing number zero. These are situations where one of the bounds from Lemmas 3.1 or 3.2 are tight.

Lemma 3.3. *Let $D = \sum_{i=1}^{m} P_{k_i}$ where $k_1 \geq k_2 \geq \cdots \geq k_m$. If both of the following conditions hold:*

(i) $k_1 - 1 = \sum_{i=2}^{m} k_i$ *and*
(ii) $\sum_{i=3}^{m} k_i \geq \lfloor \frac{k_1+k_2-1}{3} \rfloor$

then $r(D) = 0$.

Proof. Assuming conditions (i) and (ii), we can define σ so that $S(\sigma, D)$ has $s_{2i-1} = 1$ for $i = 1, 2, \ldots, k_1$ and $s_{2j} \neq s_{2j-2}$, for $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. It is clear that $S(\sigma, D)$ does not contain a forbidden string of type I. To see that $S(\sigma, D)$ does not contain a forbidden string of type II, we note that if $s_k = s_{k-2}$, then s_{k-1} does not equal s_{k-3} or s_{k-4}. Hence by Theorem 2.2, D is a minimum feedback arc set of $T(\sigma, D)$. Since $|V(T(\sigma, D))| = |V(D)|$, we conclude that $r(D) = 0$. ■

We note that if $m = 3, k_1 = 2k + 1$, and $k_2 = k_3 = k$, where k is an integer greater than or equal to 2, then we have the digraphs described in Theorem 2.1. Applying Lemma 3.3 to this family of digraphs completes the proof of Theorem 2.1.

The next lemma is the analog of Lemma 3.3 where we have equality in (ii). We note that the case where equality holds in both (i) and (ii) is covered in Lemma 3.3, so we will consider the case with strict inequality in (i).

Lemma 3.4. *Let $D = \sum_{i=1}^{m} P_{k_i}$ where $k_1 \geq k_2 \geq \cdots \geq k_m$. If both of the following conditions hold:*

(i) $k_1 - 1 < \sum_{i=2}^{m} k_i$ *and*
(ii) $\sum_{i=3}^{m} k_i = \lfloor \frac{k_1+k_2-1}{3} \rfloor$

then $r(D) = 0$.

Proof. We will use R to denote $\sum_{i=3}^{m} k_i$. If $R = 0$, then $k_1 + k_2 \leq 3$. Then $k_1 < k_2 + 1 \Rightarrow k_1 = k_2$. If $k_1 = 1$ then $D = 2P_1$, which clearly has reversing number zero. Hence, it will proceed under the assumption that $R \geq 1$. We note that (ii) enables us to define a particular ordering σ where any segment containing only vertices from P_{k_1} and P_{k_2} has at most three vertices. Let σ be an ordering that corresponds to the string $S(\sigma, D) = \langle S_1, v_1, S_2, v_2, S_3, \ldots, v_R, S_{R+1} \rangle$, where the substrings S_i will contain only vertices from P_{k_1} and P_{k_2}, and S_i will have three vertices for $i = 1, 2, \ldots, R$ and S_{R+1} will have 1, 2, or 3 vertices depending on $|V(D)|$. We note that $S_{R+1} \neq \emptyset$, since $|S_{R+1}| = 0 \Rightarrow k_1 + k_2 = 3R \Rightarrow \lfloor \frac{k_1+k_2-1}{3} \rfloor = R - 1$. This would contradict (ii).

We will consider the three cases where $|V(D)| \equiv 1, 2$, or 3 mod 4. We begin by describing all of the S_i's except for the last two, by letting $S_i = \langle 2, 1, 2 \rangle$ for

$i = 1, 2, \ldots, k_2 - R - 1$ and $S_i = \langle 1, 2, 1 \rangle$ for $i = k_2 - R, k_2 - R + 1, \ldots, R - 1$. For completeness we include all of the details. The number of 2's in this part of the sequence is $2(k_2 - R - 1) + (R - 1 - k_2 + R + 1) = k_2 - 2$ and the number of 1's is $(k_2 - R - 1) + 2(R - 1 - k_2 + R + 1) = 3R - k_2 - 1 = 3(R - 1) - (k_2 - 2)$. Next, we will describe S_R and S_{R+1} for the different possibilities for $|V(D)|$. If $|V(D)| \equiv 1 \bmod 4$ then let $S_R = \langle 2, 1, 2 \rangle$ and $S_{R+1} = \langle 1 \rangle$ and then the sequence will contain k_2 2's and $3R + 1 - k_2$ 1's. If $|V(D)| \equiv 2 \bmod 4$, then let $S_R = \langle 1, 2, 1 \rangle$ and $S_{R+1} = \langle 1, 2 \rangle$ and hence the sequence will contain k_2 2's and $3R + 2 - k_2$ 1's. If $|V(D)| \equiv 3 \bmod 4$ then let $S_R = \langle 1, 2, 1 \rangle$ and $S_{R+1} = \langle 1, 2, 1 \rangle$. Then the sequence will contain k_2 2's and $3R + 3 - k_2$ 1's.

Finally, we will show that for any of the three cases, $S(\sigma, D)$ does not contain a forbidden string. Clearly $S(\sigma, D)$ does not contain a forbidden string of type I. We next note that there is no substring of $S(\sigma, D)$ containing four consecutive terms that are either 1 or 2. Hence if $s_j = s_{j-2}$ then $s_{j-1} \neq s_{j-3}$. If $s_j = s_{j-2}$ and $s_{j-1} = s_{j-4}$ for some $j, 6 \leq j \leq n$, then it must be that s_j, s_{j-2}, s_{j-1}, and s_{j-4} are all less than or equal to 2, since two terms that are greater than 2 have at least three terms less than or equal to 2 between them. Now if $s_{j-3} \leq 2$ then $\langle s_{j-3}, s_{j-2}, s_{j-1}, s_j \rangle$ would form a substring of $S(\sigma, D)$ containing four consecutive terms that are either 1 or 2. Hence $s_{j-3} > 2$, and s_{j-3} cannot equal s_{j-5} or s_{j-6}. Thus $S(\sigma, D)$ does not contain a forbidden string of type I or II, and it follows that D is a minimum feedback arc set of $T(\sigma, D)$. Hence $r(D) = 0$. ∎

We can now combine the results up to this point to classify all digraphs that are a disjoint union of directed paths and have reversing number zero.

Theorem 3.1. *Let $D = \sum_{i=1}^{m} P_{k_i}$, where $k_1 \geq k_2 \geq \cdots \geq k_m$. Then $r(D) = 0$ if and only if both of the following conditions hold:*

(i) $k_1 - 1 \leq \sum_{i=2}^{m} k_i$ *and*
(ii) $\sum_{i=3}^{m} k_i \leq \lfloor \frac{k_1 + k_2 - 1}{3} \rfloor$.

Proof. The necessity of (i) and (ii) follows from Lemmas 3.1 and 3.2.

Let $D = \sum_{i=1}^{m} P_{k_i}$ with $|V(D)| = n$. If either (i) or (ii) hold for D with equality, then we are done by Lemmas 3.3 and 3.4. Hence we will assume that we have strict inequality for both (i) and (ii).

To prove sufficiency, we will proceed by induction on n. We have the following base cases. Clearly $r(P_1) = r(2P_1) = r(3P_1) = 0$ and it was shown in the proof of Lemma 3.3 that $r(P_2 + P_1) = 0$.

Let $D = \sum_{i=1}^{m} P_{k_i}$ and remove one vertex from each of the three largest paths, to form the digraph $D^* = \sum_{i=1}^{m} P_{k_i^*}$.

We will assume that (i) and (ii) hold for D with strict inequality and show that (i) and (ii) hold for D^*. This will imply the existence of some ordering σ^* such that D^* is a minimum feedback arc set of $T(\sigma^*, D^*)$. Then we will apply Lemma 2.2 to complete the proof.

We will consider four cases. For completeness we give the details. To simplify notation we will let $R = \sum_{i=3}^{m} k_i$ and $R^* = \sum_{i=3}^{m} k_i^*$.

Case 1. $k_1^* = k_1 - 1, k_2^* = k_2 - 1$, and $R^* = R - 1$.

First we show (i) holds. $k_1 - 1 < \sum_{i=2}^{m} k_i \Rightarrow k_1^* \leq k_2^* + R^* + 1 \Rightarrow k_1^* - 1 \leq k_2^* + R^*$ as desired. Next we show (ii) holds. $R > \lfloor \frac{k_1+k_2-1}{3} \rfloor \Rightarrow R^* + 1 > \lfloor \frac{k_1^*+k_2^*+1}{3} \rfloor \Rightarrow R^* \geq \lfloor \frac{k_1^*+k_2^*-1}{3} \rfloor$.

Case 2. $k_1^* = k_1, k_2^* = k_2 - 1$, and $R^* = R - 2$.

Since we removed a vertex from each of the three largest paths, none of which were on the path P_{k_1}, then $m \geq 4$ and $k_1 = k_2 = k_3 = k_4$. First we show that (i) holds. Since $m \geq 4$ it follows that $R \geq 2$. Then $R + k_2 \geq k_1 + 2 \Rightarrow (R-2) + (k_2 - 1) \geq k_1 - 1 \Rightarrow R^* + k_2^* \geq k_1^* - 1$.

Next, we show that (ii) holds. Since $k_1 = k_2 = k_3 = k_4, R \geq k_1 + k_2 \Rightarrow R^* + 2 \geq k_1^* + k_2^* + 1 \geq \lfloor \frac{k_1^*+k_2^*+1}{3} \rfloor$.

Case 3. $k_1^* = k_1 - 1, k_2^* = k_2$, and $R^* = R - 2$.

Since we removed a vertex from each of the three largest paths, we must have that $k_2 = k_3 = k_4$. First we show that (i) holds. $k_1 - 1 < \sum_{i=2}^{m} k_i \Rightarrow k_1^* < k_2^* + R^* + 2 \Rightarrow k_1^* \leq k_2^* + R^* + 1 \Rightarrow k_1^* - 1 \leq k_2^* + R^*$. We will use this last inequality and also the fact that $R^* + 2 = R \geq 2k_2 = 2k_2^*$ to show that (ii) holds. We then can conclude $R^* = \lfloor \frac{3R^*+2}{3} \rfloor \geq \lfloor \frac{2R^*+2k_2^*}{3} \rfloor \geq \lfloor \frac{R^*+(k_1^*-k_2^*-1)+2k_2^*}{3} \rfloor \geq \lfloor \frac{k_1^*+k_2^*-1}{3} \rfloor$.

Case 4. $k_1^* = k_1, k_2^* = k_2$, and $R^* = R - 3$.

Since we removed a vertex from each of the three largest paths, we must have that $k_1 = k_2 = k_3 = k_4 = k_5$. Since $R^* \geq 1$ and $k_2^* = k_1^*$, clearly we must have $k_1^* \leq \sum_{i=2}^{m} k_i^*$. Next we show that (ii) holds. If $k_1 = 1$, then D consists of isolated vertices and the result is trivial. If $k_1 \geq 2$ then $3R \geq 3\sum_{i=3}^{5} k_i \Rightarrow 3R^* + 9 \geq 3k_1^* + 3k_2^* + 6 \Rightarrow 3R^* \geq 3k_1^* + 3k_2^* - 3 \Rightarrow R^* \geq \lfloor \frac{k_1^*+k_2^*-1}{3} \rfloor$. We can then conclude that D^* is a minimum feedback arc set of $T(\sigma^*, D^*)$.

Let v_a, v_b, and v_c be vertices in $V(D) \backslash V(D^*)$ each on different directed paths. Without loss of generality, we will assume that v_a is a vertex not on the same path as $\sigma^{-1}(|V(D^*)|)$ or $\sigma^{-1}(|V(D^*)| - 1), v_b$ is not on the same path as v_a or $\sigma^{-1}(|V(D^*)|)$, and v_c is not on the same path as v_a or v_b. Then let σ be an ordering of $V(D)$, where $\sigma(v_a) = n - 2, \sigma(v_b) = n - 1, \sigma(v_c) = n$, and $\sigma(v) = \sigma^*(v)$ for all $v \in V(D^*)$. Then we apply Lemma 2.2 three successive times (once for each of the vertices a, b, and c) to conclude that D is a minimum feedback arc set of $T(\sigma, D)$. Finally, since $V(T(\sigma, D)) = V(D)$, we have that $r(D) = 0$. ■

B. Reversing Number for a Disjoint Union of Directed Paths

We now have the tools to precisely determine the reversing number for a disjoint union of directed paths, our main result of the section.

Theorem 3.2. *Let $D = \sum_{i=1}^{m} P_{k_i}$, where $k_1 \geq k_2 \geq \cdots \geq k_m$. Then*

$$r(D) = \max\left\{0, k_1 - 1 - \sum_{i=2}^{m} k_i, \left\lfloor \frac{k_1 + k_2 - 1}{3} \right\rfloor - \sum_{i=3}^{m} k_i \right\}.$$

Proof. Let $|W| = \max\{0, k_1 - 1 - \sum_{i=2}^{m} k_i, \lfloor\frac{k_1+k_2-1}{3}\rfloor - \sum_{i=3}^{m} k_i\}$. First we note that if $|W| = 0$ then $k_1 - 1 \leq \sum_{i=2}^{m} k_i$ and $\lfloor\frac{k_1+k_2-1}{3}\rfloor \leq \sum_{i=3}^{m} k_i$. Then by Theorem 3.1, we have that $r(D) = 0$.

Now let $|W| > 0$. By Theorem 3.1, we have that $r(D + |W|K_1) = 0$ and then by Lemma 1.4, $r(D) \leq |W|$. Next, we seek to show that $r(D) \geq |W|$. Assume that $r(D) = |F| < |W|$. Then by Lemma 1.4, $r(D + |F|K_1) = 0$ and by Theorem 3.1 both of the following conditions hold:

(i) $k_1 - 1 \leq (\sum_{i=2}^{m} k_i) + |F|$ and
(ii) $\lfloor\frac{k_1+k_2-1}{3}\rfloor \leq (\sum_{i=3}^{m} k_i) + |F|$.

Then $|F| \geq k_1 - 1 - \sum_{i=2}^{m} k_i, |F| \geq \lfloor\frac{k_1+k_2-1}{3}\rfloor - \sum_{i=3}^{m} k_i$ and $|W| > |F|$ imply $|W| > k_1 - 1 - \sum_{i=2}^{m} k_i$ and $|W| > \lfloor\frac{k_1+k_2-1}{3}\rfloor - \sum_{i=3}^{m} k_i$.

Hence $|W| > \max\{0, k_1 - 1 - \sum_{i=2}^{m} k_i, \lfloor\frac{k_1+k_2-1}{3}\rfloor - \sum_{i=3}^{m} k_i\}$, a contradiction. ■

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