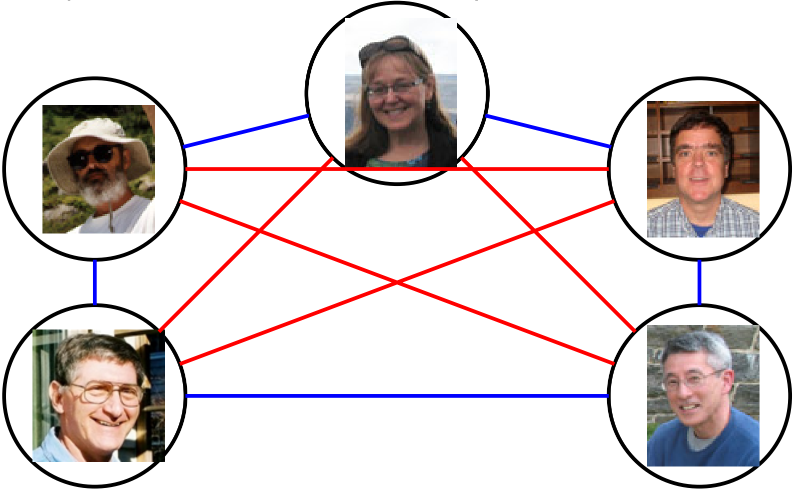
Boundaries for Graph Ramsey Numbers

Garth Isaak
parts from Jonelle Hook Dissertation
Lehigh University

Show that every group of 6 facebook users includes:
a clique of 3 mutual friends OR a clique of 3 mutual nonfriends



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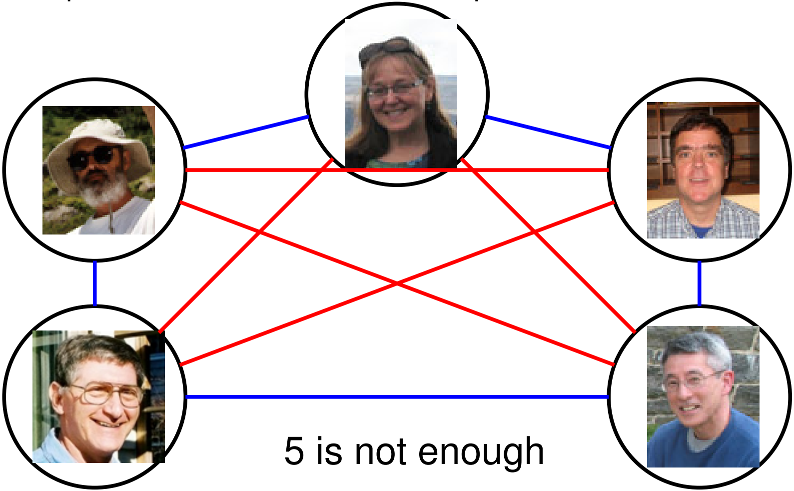

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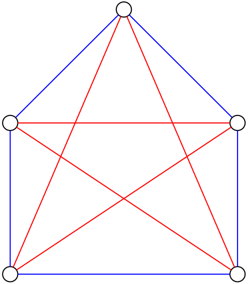

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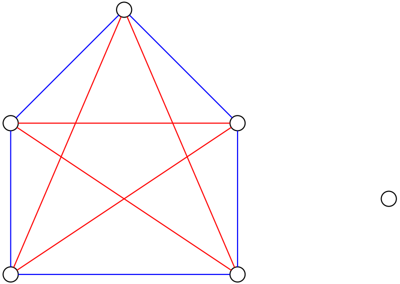
‘Proof’ that every 2 edge coloring of complete graph on 6 vertices contains a blue or red 3-clique:



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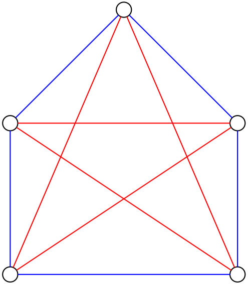

If the 6th person has 3 friends
2 must be adjacent
similarly if 3 nonfriends

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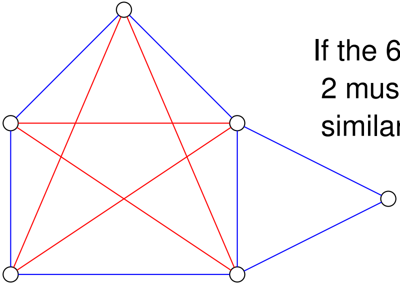

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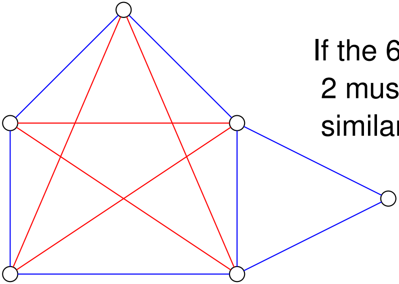

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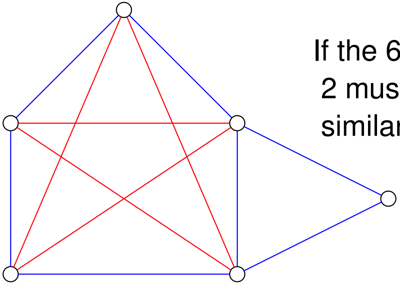

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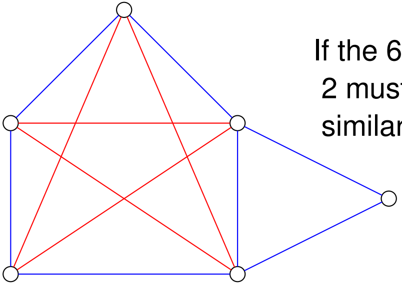

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Incorrect ‘proof’ will motivate later ideas

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- Barack has 3 friends or 3 nonfriends
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Notation For Ramsey Numbers by example:

- $R(3,3) = 6$ *because*
- *Every group of 6 people has a clique of 3 friends or a clique of 3 nonfriends*
- *Some group of 5 fails to have this property*

- $R(4,2) = 4$ *because*
- *Every group of 4 people has a clique of 4 friends or a clique of 2 nonfriends*
- *Some group of 3 fails to have this property*

Show that $R(4,3) \leq 10$

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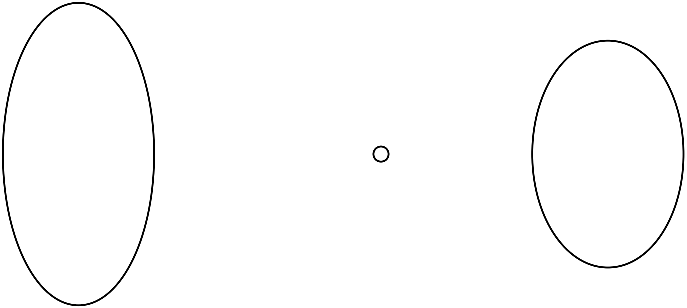
- Barack either has 6 friends or 4 nonfriends
-
-



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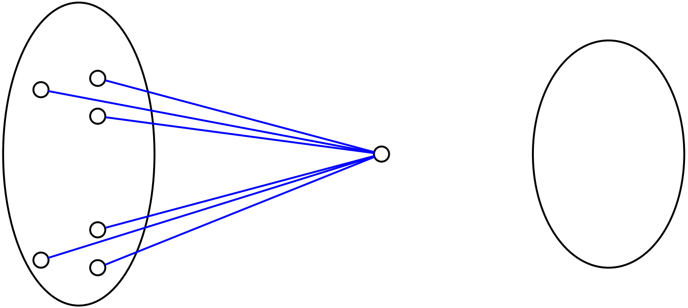

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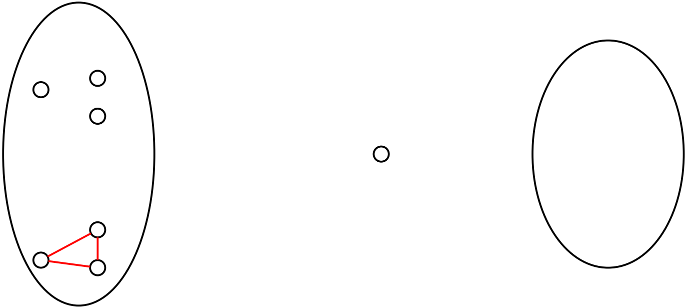
- 

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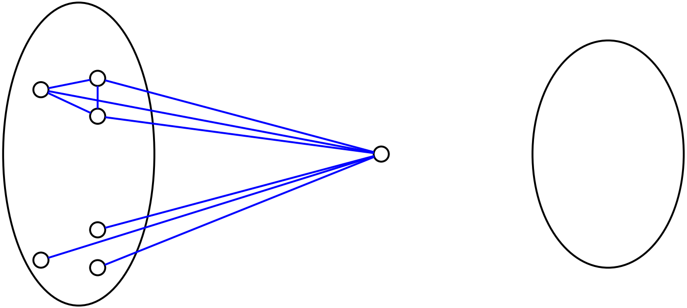

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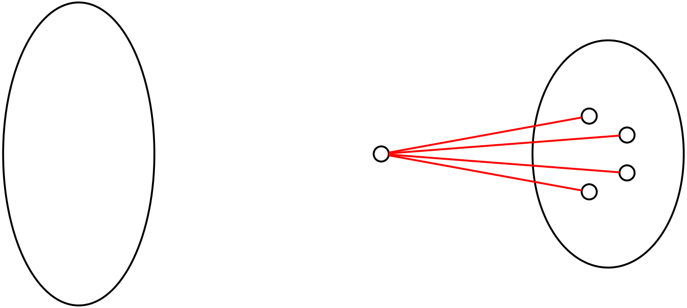

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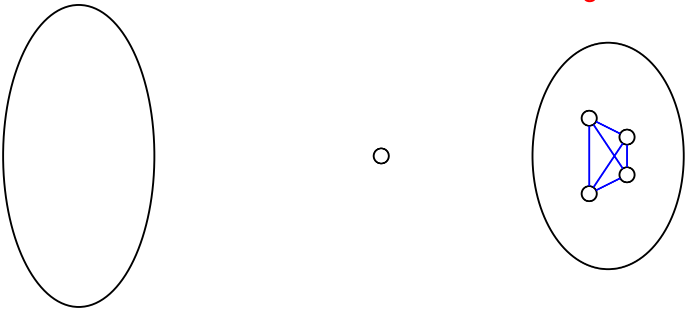

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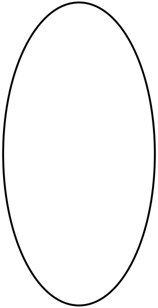

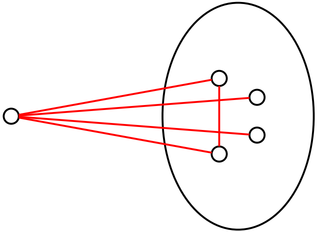

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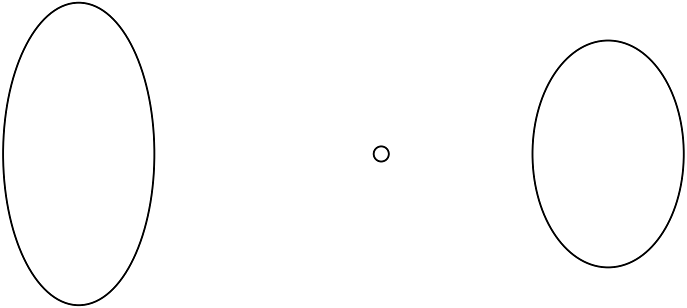

$R(4,3) \leq 9$: by parity someone has 6 friends or 4 nonfriends

- In general
  $R(p,q) \leq R(p-1,q) + R(p,q-1)$
- 'Same' recursion as Binomial (Pascal's) triangle:
- $\Rightarrow \cdots \; R(p,q) \leq \binom{p+q-2}{p-1}$

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How good is this?

Bound implies $R(5,5) \leq 70$

With parity can improve to ≤ 62

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In fact $R(5,5)$ is 43 or 44 or $\cdots$ or 49

We don't actually know

Known Ramsey numbers

$R(2, n) = n$ and

m	n	$R(m, n)$	ref
3	3	6	Greenwood & Gleason 1955
3	4	9	Greenwood & Gleason 1955
3	5	14	Greenwood & Gleason 1955
3	6	18	Graver & Yackel 1968
3	7	23	Kalbfleisch 1966
3	8	28	McKay & Min 1992
3	9	36	Grinstead & Roberts 1982
4	4	18	Greenwood & Gleason 1955
4	5	25	McKay & Radiszowski 1995

Just use a big computer to find $43 \leq R(5,5) \leq 48$

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Try checking ALL possible sets of relationships with 49?

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All of the atoms in the known universe checking a billion possibilities per second

Try checking <u>ALL</u> possible sets of relationships with 49?

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Still not done checking all possibilities

- Can't find exact Ramsey numbers so ...
- Approximate Ramsey numbers for large clique sizes
-
-
-

- Can't find exact Ramsey numbers so ...
- Approximate Ramsey numbers for large clique sizes
- Binomial upper bound $\Rightarrow R(n, n) < 4^n$
- Lower bound from Erdos 1947 introduced probabilistic method: $R(n, n) \geq (\sqrt{2})^n$
- $\left(\sqrt{2}\right)^n < R(n, n) < 4^n$

Illustrate lower bound method by showing $R(5,5) > 11$

- There are $\binom{11}{5} = 462$ groups of 5 among 11 people
- $\binom{5}{2} = 10$ relationships in a group of 5
- Probability that all friends or all nonfriends $2 \cdot \left(\frac{1}{2}\right)^{10} = \frac{1}{512}$
- Ignore dependencies among groups:
  Probability some group all friends or all nonfriends
  $= 462 \cdot \frac{1}{512} < 1$
- Random relationships $\Rightarrow$ positive probability no group of 5 all friends or nonfriends
- For large values $\Rightarrow \sqrt{2}^n < R(n,n)$

Frank Plumpton Ramsey (1903 - 1930)

 

Few philosophers of the twentieth century have influenced the sciences as much as Ramsey. He did pioneering work in pure mathematics, logic, economics, statistics, probability theory, decision theory and cognitive psychology. He also did ground-breaking work on epistemology, philosophy of science, philosophy of mathematics, metaphysics and semantics. And he accomplished all this before the age of twenty-seven.

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Ramsey's statement of his lemma:

Let G be an infinite class, and m and r positive integers; and let all those sub-classes of G which have exactly r members, or, as we may say, let all r-combinations of the members of G be divided in any manner into m mutually exclusive classes $C_i (i = 1, 2, \ldots, m)$, so that every r-combination is a member of one and only one C_i; then, assuming the Axiom of Selections, G must contain an infinite sub-class D such that all the r-combinations of the members of D belong to the same C_i.

Other variants

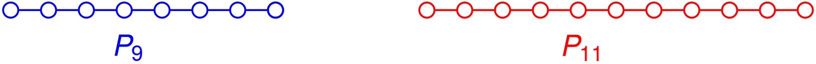
- *Schur 1916: k-color enough integers get monochromatic $x + y = z$*
- *Van Der Waerden 1927 : k-color enough integers get long monochromatic arithmetic progressions*

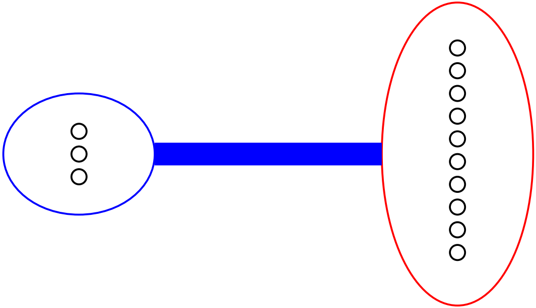
So we can't determine Ramsey numbers exactly, what might we do?

- *Asymptotics for general versions (multiple colors, hyperedges)*
- *......*
- *Graph Ramsey numbers (next)*

Look at 'easier' variant that uses less machinery

Graph Ramsey Theory: $R(P_9, P_{11}) > 13$

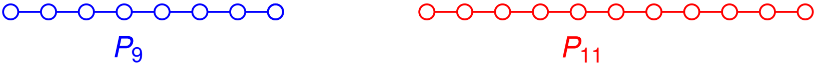


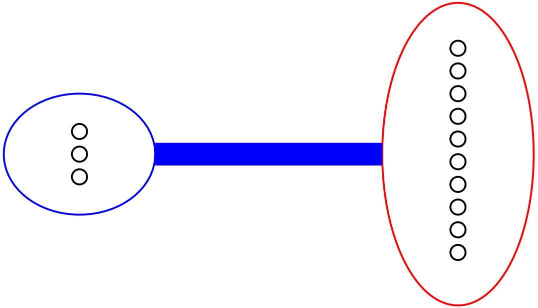


$R(P_n, P_m) = m + \lfloor \frac{n}{2} \rfloor - 1$ for $2 \leq n \leq m$

(Gerencser and Gyarfas 1967)

$R(P_9, P_{11}) > 13$

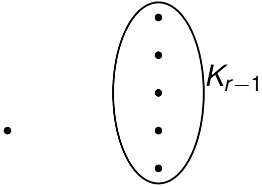




14th person arrives.

Add relationships one at a time, when is P_9 or P_{11} forced?
There are other ‘critical graphs’ with 13 vertices

Star critical Ramsey:
$R(G, H) = r$ add/color edges to K_{r-1} one at a time:
When is a red G or blue H forced?



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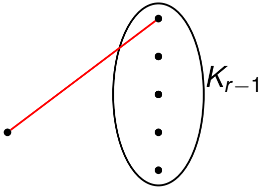

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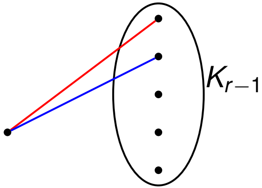

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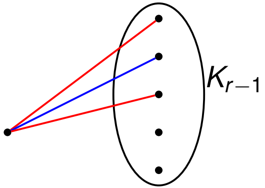

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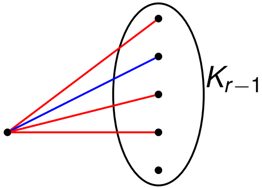

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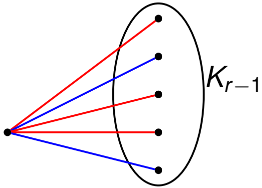

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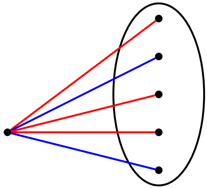

- *Proofs: First classify sharpness examples*
  Good colorings of K_{r-1}
- *Examples with 'few' extra edges needed and with 'many' extra edges needed*

Example

- $R(K_m, K_n) = r$: *must add all* $r - 1$ *edges (Chvatal 1974) even though we do not know what* r *is*
-
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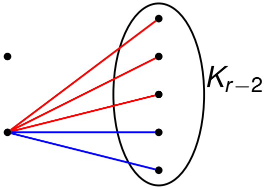

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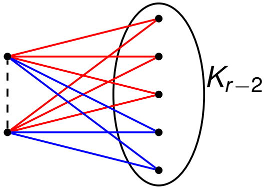

Example

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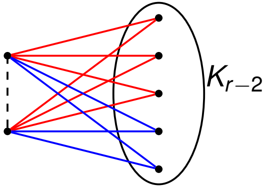
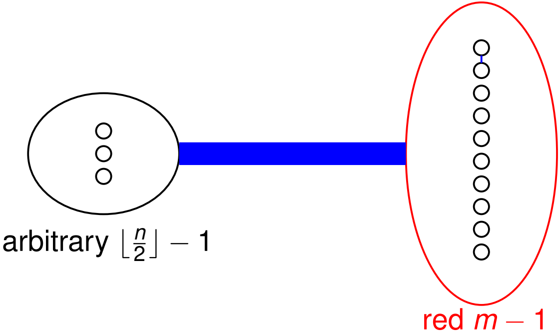
- *make a copy of a vertex*
- *similar for* $R(mK_3, mK_3) = 5m$



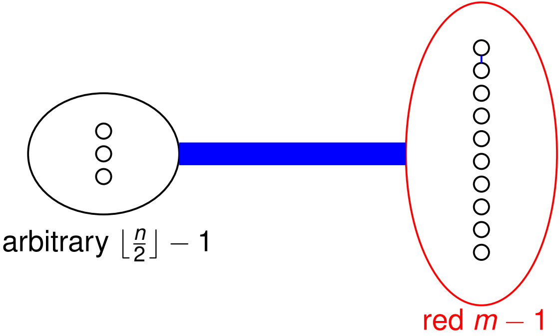
$$R(P_n, P_m) = m + \lfloor \tfrac{n}{2} \rfloor - 1 \text{ for } 2 \le n \le m$$

- 'blue' graph on left can be arbitrary
- allow one blue edge on right when n is odd
- other cases when $m = n$ or $m = n + 1$



$$R(P_n, P_m) = m + \lfloor \frac{n}{2} \rfloor - 1 \text{ for } 2 \leq n \leq m$$

- Classification of critical graphs proof:
- If everyone has 10 friends then path of 21 friends
- induction using also path, cycle results, multiple cases



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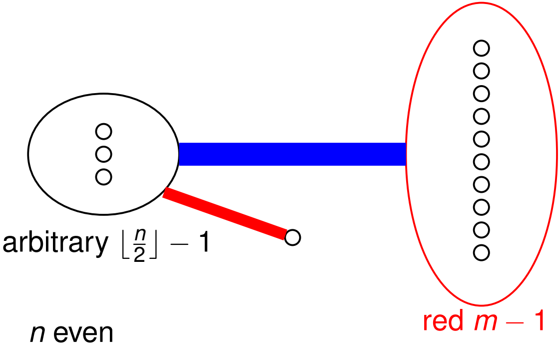
Star critical Ramsey number is $\lceil \frac{n}{2} \rceil$



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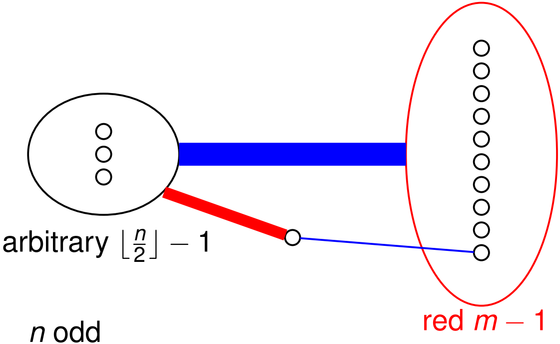

n odd

Summary of some Results for known graph Ramsey numbers

Ramsey number	R^*
$R(T_n, K_m) = (n-1)(m-1)+1$	$(n-1)(m-2)+1$
$R(C_n, K_4) = 3n-2$	$2n$
$R(P_n, P_m) = n + \lfloor m/2 \rfloor - 1 \; n \geq m$	$\lceil m/2 \rceil$
$R(P_n, C_m) = 2n-1 \; n \geq m$, m odd	n
$R(P_n, C_m) = n + m/2 - 1 \; n \geq m$, m even	$m/2+1$
$R(F_n, K_3) = 4n+1$	$2n+2$
$R(K_n, mK_2) = n+2m-1$	$2n+2$
$R(nK_4, mK_3) = 4n+2m+1$, $n \geq m$	$4n+2m$
$R(nK_4, mK_3) = 3n+3m+1$, $m \geq n$	$3n+3m$

1st half Hook 2010

2nd half Li and Li 2013

Problems

- There are many known (exact values) Graph Ramsey numbers
  usually for highly structured graphs - classify critical graphs and find star critical number.
- Still many graph Ramsey number problems.
- Extend to more colors (even fewer known values)