Cycle Extendability in Ptolemaic Graphs

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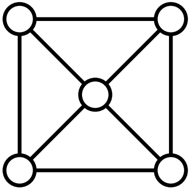
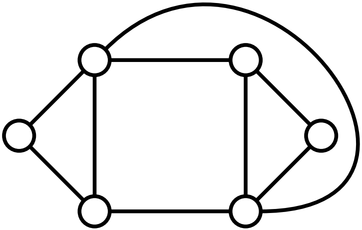
Chordal Graph

'=' Triangulated
'=' Every cycle has a chord

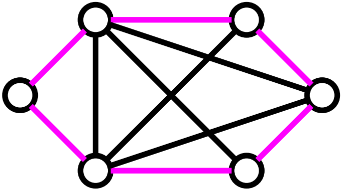
Not Chordal



Chordal



Simplicial vertex - Neighborhood is a clique
Chordal graphs have them
Hamiltonian chordal graph
‘Peel off’ simplicial vertices to get cycles of all sizes.



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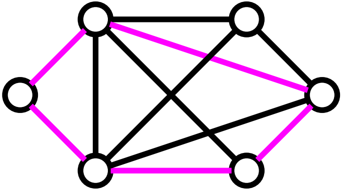

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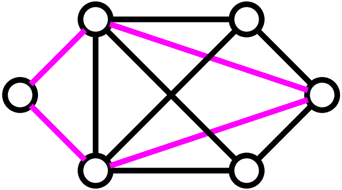

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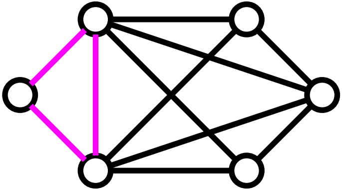

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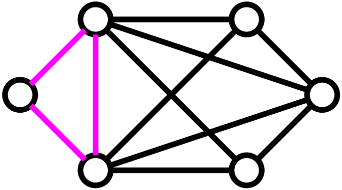
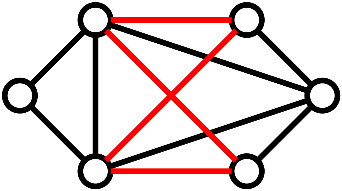


Hendry (1990) conjecture - This process can be reversed in Hamiltonian Chordal Graphs starting with any cycle (sort of)
'Exact' reverse would imply polynomial algorithm for Hamiltonian cycles in chordal graphs but its NP-hard even on chordal graphs.

‘Exact’ reverse fails

Instead C extends if

- G - Hamiltonian and Chordal
- C - cycle in G
- For <u>some</u> x there is <u>some</u> cycle on $V(C) + x$



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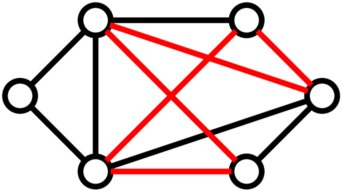

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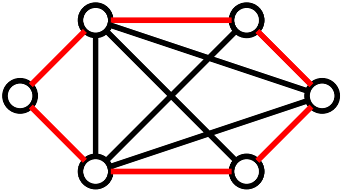

Conjecture (Hendry 1990)

Cycles in Hamiltonian chordal graphs can be extended

Shown for several subclasses

- T. Jiang (2002) Planar Chordal
- Chen, Faudree, Gould, Jacobson (2006) interval graphs
- Abeuida and Sritharan (2006) interval graphs plus ...

Our Result

Conjecture holds for Ptolemaic graphs

G is Ptolemaic

$\Updownarrow$

Ptolemaic inequality holds

$\Updownarrow$

G is distance hereditary and chordal

$\Updownarrow$

$\vdots$

$\Updownarrow$

For all distinct nondisjoint maximal cliques $M_1 \cap M_2$ separates $M_1 - M_2$ from $M_2 - M_1$

G Ptolemaic:
For all distinct nondisjoint maximal cliques
$M_1 \cap M_2$ separates $M_1 - M_2$ from $M_2 - M_1$

Useful fact

G Hamiltonian, v simplicial then $G - v$ is Hamiltonian

Proof idea

1. If Cycle C avoids simplical vertex v
   Extend C in $G - v$ by induction
2. If Cycle C contains all simplicial vertices:

Proposition

Every vertex not on C is adjacent to an edge of C

'Reverse' peeling off process

Proposition

G Ptolemaic
Cycle C contains all simplicial vertices
Every vertex not on C is adjacent to an edge of C

Useful fact

G Ptolemaic
M, M_1, M_2 maximal cliques
$S_i = M \cap M_i$
$\Updownarrow$
$S_1 \cap S_2 = \emptyset$ or $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$

in part from result of Uehara and Uno 2005:

Proposition

M a maximal clique in Ptolemaic G partitions into separators S_i as ...

Proposition

G Ptolemaic
Cycle C contains all simplicial vertices
Every vertex not on C is adjacent to an edge of C

Proof idea: Take maximal clique M containing v
C must contain and edge of M to get to simplicial vertices at ‘ends’ of ‘pieces’