Degree Sequences for Multitrees

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Abstract

Two basic exercises in graph theory are characterizing degree sequences of trees and degree sequences of multigraphs. For each there are several proofs using different approaches. The goal here is to examine the interesting exercises that arise when we combine these two problems by looking at degree sequences of multigraphs with an underlying forest like structure.

1 Basics

A quick review of graph theory terminology and notation will appear at the end of this section. For now we assume familiarity. The basic results and definitions that we refer to can be found in almost any graph theory text. Two basic exercises in an introductory graph theory course are to characterize those sequences of positive integers that can occur as degrees of a tree or as degrees of a multigraph. For each there are, at least, several different approaches to a proof. Our aim here is to explore what happens when we combine these and look at multigraphs with underlying tree structure. Similarities in one standard inductive proof for each of the problems lead us to both a nice example of why we care about a basis in induction and to see how multigraphs have realizations with underlying graph 'close' to a tree. As we explore these variations we get a variety of problems with proofs that are straightforward enough that they could be posed as additional exercises in introductory graph theory courses.

The basic results that we observe include:

- The degrees of multigraphs can be realized with underlying graph a forest or a unicyclic graph where the unique cycle has length 3.

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- Multigraphs can be realized with underlying graph a forest if and only if the degree sequence partitions into two parts with equal sums. In particular the degrees of a bipartite multigraph can be realized with underlying graph a forest.
- The degrees of a multigraph with an underlying forest realization have a connected (i.e., an underlying tree) realization if and only if the degree sum divided by the greatest common divisor of the degrees is large enough.

As most of the results we observe are either already well known or reasonably straightforward generalizations of such known results, we will provide proof sketches rather than detailed proofs for our claims. For some of these we will also include a 'proof by example' as an illustration. Our goal will also be to examine different approaches to proving these results. The first and second results a slight refinements of those found in [3] with slightly different terminology and proofs along the lines of our Proof 2 of Fact 8. The result in [3] involves the structure of realizations that minimize the number of underlying edges and shows that each component has an odd cycle or no cycles but the technique easily would give length 3 as we do here. The results reviewed in Sections 2 and 3 are well known and can be found in most introductory graph theory texts.

For completeness, we include a quick review of graph theory terminology. All graphs and sequences that we consider will be finite. A graph $G = (V, E)$ is a set V of vertices along with a set E of edges which are pairs of distinct vertices. A loop would be an edge in which both vertices are the same. None of the graphs that we consider here will have loops. A multigraph allows E to be a multiset. The degree of a vertex is the number of edges that contain it and the degree sequence is the sequence of degrees. A realization of a degree sequence is a graph with the given degrees. As each edge contains two vertices, the degree sequence sums to twice the number of edges and hence has even sum. This is often called the handshaking lemma.

A path in a graph is a set of edges of the form $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{t-1}, v_t\}$ with the v_i distinct. We use the notation $v_1, v_2, \ldots, v_t$. A cycle is a set of edges as in a path except that $v_t = v_1$. By this definition a pair of parallel edges $\{v_1, v_2\}, \{v_2, v_1\}$ could be considered a 2-cycle but we will not call this a cycle. A forest is a graph with no cycles. A graph is connected if there is a path between any pair of vertices. The components of a disconnected graph are the maximal connected subgraphs. A tree is a connected forest. A unicyclic graph is a graph with exactly one cycle, deleting any edge on this cycle results in a forest.

The underlying graph of a multigraph is the graph obtained by keeping one copy of each multiple edge. Multitrees and multiforests are multigraphs with underlying graphs that are trees and forests respectively.

We will use the fact that the degree of a vertex on a cycle is at least 2 in the form that a vertex of degree 1 (called a leaf) is on no cycle.

Characterizing which integer sequences are degree sequences of some graph is, somewhat surprisingly, not immediately obvious. There are at least a half dozen variants on how to state the conditions and many different versions of proofs have been and continue to be given. None of these are very short. A careful treatment presented takes up a significant portion of an hour long undergraduate class and written fills at least a page. On the other hand, characterizing degree sequences for certain classes of graphs yield very short proofs with some insights for beginning students as well as the opportunity for exercises asking for alternate proof approaches. We concern ourselves here with two well known instances of these, trees and multigraphs, and then explore some extensions that combine these two classes. See Figure 1 for an example of a multitree and its degree sequence.

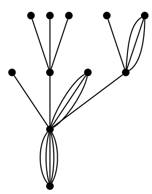


Figure 1: A multitree with degrees (11,5,5,4,3,3,1,1,1,1,1)

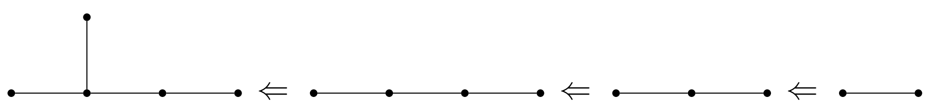
2 Forests and trees

Some basic facts about trees (that are themselves good introductory exercises) are that every tree (except the trivial tree consisting of a single vertex and no edges) has at least two vertices with degree 1 (indeed the number of leaves is at least the maximum degree), and that the number of edges is one less than the number of vertices. So, a tree with n vertices has $n-1$ edges. As the degree sum is twice the number of edges it must be $2n-2$.

Question 1 *When is a positive integer sequence with even sum the degree sequence of a tree?*
A necessary condition for a sequence with n terms is that the sum is $2n-2$.
Is this sufficient?

We observed a simple necessary condition above. This obvious necessary condition turns out to be sufficient. One standard proof builds on the idea that trees have leaves: note that the degree sum condition implies that some term is exactly 1 and another is at least 2 (if there are at least 3 terms). Reduce one number that is at least 2 in the potential degree sequence by 1 and remove a 1. By induction on the new sequence, form a tree then add an edge joining a new leaf to a vertex with the reduced degree. One end of the new edge has degree 1, hence the edge is not on a cycle and adding it has not created a cycle. So we get a tree. The basis is the sequence $1, 1$ realized by an edge. If we allow degree sums less than $2n - 2$ then inductively we reach a sequence $1, 1, \ldots, 1$ which can be realized as a set of disjoint edges and hence is not connected. So we get a forest but not a tree for smaller degree sums. We have just outlined one of several approaches to prove the following well known fact.

Fact 2 *Positive integers $d_1, d_2, \ldots d_n$ with even sum are the degrees of a forest if and only if $d_1 + d_2 + \cdots + d_n \leq 2n - 2$. There is a tree (connected forest) with these degrees if and only if equality holds.*



$(3, 2, 1, 1, 1) \quad \Rightarrow \quad (2, 2, 1, 1, \ ) \quad \Rightarrow \quad (1, 2, 1, \ , \ ) \Rightarrow \quad (1, 1, \ , \ , \ )$

Figure 2: Illustrating a tree degree recursion

In Figure 2 we illustrate the recursive construction for tree degrees implied by the inductive proof.

3 Multigraphs

As we do not allow loops in multigraphs, a vertex with largest degree must have another vertex at the other end of each edge. So the largest degree is at most the sum of the other degrees. This is a necessary condition for degree sequences of multigraphs.

Question 3 *When is a a nonnegative integer sequence with even sum the degree sequence of a multigraph?*
A necessary condition is that the largest term is at most the sum of the others.
Is this sufficient?

We observed a simple necessary condition above. This, along with the basic condition that the degree sum must be even turns out to be sufficient.

To illustrate one of the standard proofs for sufficiency of the multigraph conditions we need a little notation. We want to show that a sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ of nonnegative integers with even sum is the degree sequence of some multigraph if $d_1 \leq d_2 + \cdots + d_n$. Consider the sequence $d_1 - d_n, d_2, \ldots, d_{n-1}$. If the largest term in the new sequence is at most the sum of the others, form a multigraph by induction and add d_n edges joining a new vertex with a vertex of degree $d_1 - d_n$. This includes the trivial case $d_n = 0$.

So we need to check if the new sequence satisfies the condition on the largest term (noting also that it is clear that the new sequence has even sum).

If $d_1 - d_n$ is largest then

$$d_1 \leq d_2 + \cdots + d_n \Rightarrow d_1 - d_n \leq d_2 + \ldots + d_{n-1}$$

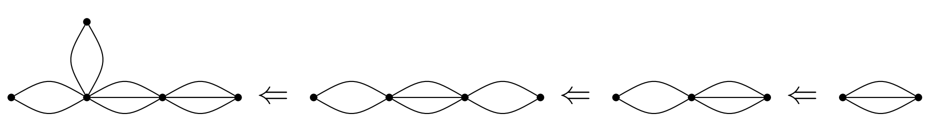
If d_2 is largest then

$$d_2 \leq d_1 \text{ and } d_n \leq d_{n-1} \Rightarrow d_2 + d_n \leq d_1 + d_{n-1} \Rightarrow d_2 \leq (d_1 - d_n) + d_3 + \cdots + d_{n-1}$$

Hence the condition holds for the new sequence and we complete the proof by induction. We have just outlined one of several approaches to prove of the following well known fact.

Fact 4 *Nonnegative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with even sum are the degrees of a multigraph if and only if $d_1 \leq d_2 + \cdots + d_n$.*

In Figure 3 we illustrate the recursive construction for multitree degrees implied by the inductive proof.



$(7,5,2,2,2) \Rightarrow (5,5,5,2,\ ) \Rightarrow (3,5,2,\ ,\ ) \Rightarrow (3,3,\ ,\ ,\ )$

Figure 3: Illustrating a multitree degree recursion

4 Near Multiforests

If we look at the proof outlines for degree sequences of forests and for degree sequences of multigraphs we note a great deal of similarity. In each case we 'removed' a vertex of minimum degree d_n and 'removed' d_n edges from a vertex of larger degree, used induction and then added d_n edges between these two vertices. While this description gets the point across it also illustrates a possible error made by beginning students. We are handed a sequence of integers, so we can't 'remove' vertices and edges. Instead we reduce the values in the sequence, use induction to get a graph that does have vertices and edges, and then add a vertex and edges to get the right degrees.

Just as the last step in the tree proof, adding an edge to a new vertex, does not create a cycle, the last step in the multigraph proof, adding multiple edges to a new vertex does not create any cycles in the underlying graph.

It would seem then that our multigraph proof has actually shown the following.

False Claim 5 *If a nonnegative integer sequence is the degree sequence of a multigraph then it is the degree sequence of a multiforest.*

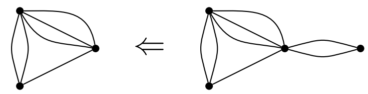
Question 6 *Why is False Claim 5 False?*

Trees are bipartite, the vertices can be partitioned into two parts such that every edge has one vertex from each part. Hence degree sequences of bipartite graphs have the additional property that they can be partitioned into two parts with equal sums. If correct, our claim would have shown that all integer sequences with even sum in which the largest value is at most the sum of the rest can be partitioned into two parts with equal sums. This is of course false, consider for example the sequence $2, 2, 2$ or slightly less trivial $5, 4, 3$.

The general problem of partitioning a sequence into two parts with equal sums is NP-hard, so if our proof was correct it would essentially imply an elementary and efficient algorithm for an NP-hard problem and imply that all sequences in which the largest value is not too large could be partitioned into two parts with equal sum.

Question 7 *What went wrong with our 'proof' of False Claim 5?*

The examples $2, 2, 2$ and $5, 4, 3$ give a hint. Note that we skipped the 'obvious' basis for our induction in the multigraph proof outline. Hidden in the 'd_2 largest' part of the multigraph proof was an assumption that our sequence have a least 4 terms. When we reduce $5, 4, 3$ as in the proof we get the sequence $2, 4$ for which that largest term is not at most the sum of the rest. Consider the example in Figure 4, the sequence we obtain from the reduction $4, 5, 3$ has the unique realization with underlying graph a 3 cycle as indicated in the figure. Hence the proof only shows that the sequence $(6, 5, 3, 2)$ has a multigraph realization. This sequence does have a multitree realization as noted later, but the proof does not show this.



$(6, 5, 3, 2) \;\Rightarrow (4, 5, 3, \;\;)$

Figure 4: Illustrating a multigraph degree recursion with basis size 3

The rest of the proof is correct so we have a nice example illustrating why we need to pay attention to the basis in induction proofs.

We can salvage the multigraph proof by providing a correct basis when $n = 3$. That the statement is correct when $n = 2$ is indeed obvious. Given a sequence $d_1 \geq d_2 \geq d_3$ with even sum and $d_1 \leq d_2 + d_3$ form the multigraph with vertices v_1, v_2, v_3 and $\frac{d_1+d_2-d_3}{2}$ edges between v_1 and v_2, $\frac{d_1+d_3-d_2}{2}$ edges between v_1 and v_3 and $\frac{d_2+d_3-d_1}{2}$ edges between v_2 and v_3. The conditions ensure that the numbers for the edges are indeed nonnegative integers so we get a multigraph with correct degrees.

The salvaged proof tells us how 'close' we can get to realizing multigraphs as forests. Our induction for the proof of Fact 4 along with the revised basis provide a proof of the following.

Fact 8 *A sequence of nonnegative integers that is the degree sequence of a multigraph is the degree sequence of a multigraph for which the underlying graph either has no cycles or is unicyclic with the cycle having size 3.*

Later we will outline several additional proof approaches to Fact 8.

We note also that the proof outlined above does not tell us when a multigraph has a mutiforest realization. We have already noted the difficulty in determining if a sequence of integers partitions into two parts with equal sums. For a given sequence realizable as a multigraph, translating the inductive proof process into a recursive algorithm can produce a realization with an underlying 3-cycle even if there is an underlying forest realization. In particular, the realization of the sequence $(6, 5, 3, 2)$ illustrated in Figure 4 has a 3-cycle while there is a realization of these degrees with an underlying path on 4 vertices by placing 3 edges between the first two pairs on the path and 2 edges between the last pair.

5 Multiforests

Fact 8 showed that multigraphs have realizations that are nearly multiforests. The question of characterizing which degree sequences can be realized with an underlying forest or tree will fall back to the fact that degree sequences of bipartite graphs can be partitioned into two parts with equal sums. This is a necessary condition as forests are bipartite. Note that we no longer need to explicitly state the even sum condition or the bound on the largest degree as the equal sums partition condition will imply these.

Question 9 *When is a a nonnegative integer sequence with even sum the degree sequence of a multiforest?*
A necessary condition is that the sequence partitions into two parts with equal sums.
Is this sufficient?

The same basic proof idea used for trees and multigraphs works here. Assume that the degree sequence can be partitioned into two parts with equal sums, $r_1, \ldots, r_a$ and $s_1, \ldots, s_b$ with $r_1 + \cdots + r_a = s_1 + \cdots + s_b$. Pick a smallest value, say s_b, consider the sequences $r_1 - s_b, r_2, \ldots, r_a$ and $s_1, \ldots, s_{b-1}$, induct and add s_b edges between a new vertex and a vertex with degree $r_1 - s_b$. In this case there are no issues with the basis. Note that this works for any partition of the degrees into parts with equal sum. We have just outlined proofs of the following.

Fact 10 *Nonnegative integers $d_1, \ldots, d_n$ are the degree sequence of a multiforest if and only if they partition into two parts with equal sums. For every partition of the d_i into two parts with equal sum there is a realization for which the degrees in the parts correspond to the partition.*

Corollary 11 *If nonnegative integers $d_1, \ldots, d_n$ are the degree sequence of a multigraph with underlying graph that is bipartite then there is a multiforest realization with the same bipartition.*

By previous comments note that this implies that testing if an integer sequence can be realized as a bipartite multigraph (equivalently as a multiforest) is NP-hard.

6 Multitrees

Having found conditions for multiforest degrees we next consider multitree degrees.

Question 12 *When is a positive integer sequence with even sum the degree sequence of a multitree?*

Determining when a multiforest has a connected realization, i.e., a realization with underlying graph that is a tree takes a little more effort. The edge multiplicities in a multigraph are the number of times each edge appears. We use $g = \gcd$ to indicate greatest common divisor. If the greatest common divisor of the degrees is $g > 1$ then in a forest realization the edge multiplicity for edges adjacent to leaves (degree 1 vertices) in the underlying graph are multiples of g. Removing all such we get a forest in which the greatest common divisor is a multiple of g. Hence inductively we get the following.

Fact 13 *If G is a multiforest with degrees $d_1, \ldots, d_n$ then all edge multiplicities are a multiple of* $\gcd(d_1, \ldots, d_n)$.

So we may as well consider the sequence $d_1/g, d_2/g, \ldots, d_n/g$ where $g = \gcd(d_1, \ldots, d_n)$. There is a realization for these 'reduced' degrees if and only if there is a realization with the original degrees with each edge multiplicity g times as big. If the sum of these values $\frac{\sum_{i\in[n]} d_i}{\gcd(d_1,\ldots,d_n)}$ is less than $2n-2$ then there cannot be a connected realization. A connected graph has at least $n-1$ edges and hence degree sum at least $2n-2$. We will see that if the sum is at least $2n-2$ then we can get a connected realization.

An approach to get a connected realization uses the idea of switching, which preserves degrees. Switching is a common technique for certain problems related to degrees in graphs. The following fact is obvious and illustrated in Figure 5.

Fact 14 *If a multigraph has edges $\{w, x\}, \{y, z\}$ then deleting these edges and adding edges $\{w, y\}, \{x, z\}$ results in a new multigraph with the same degree sequence. We call this an edge switch.*

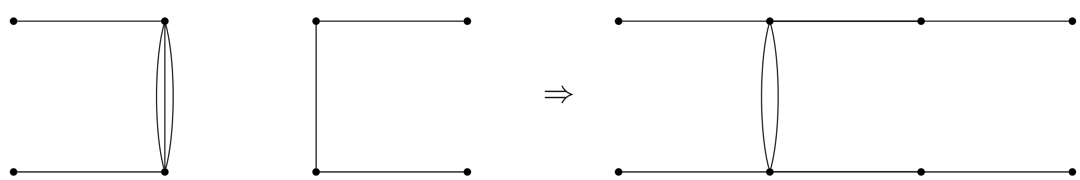


Figure 5: An example of switching

Consider a forest realization of a multigraph. For $s > r$, if edge $\{w, x\}$ with multiplicity r and edge $\{y, z\}$ with multiplicity s are in different components then removing r copies of edges $\{w, x\}, \{y, z\}$ and adding r copies of edges $\{w, y\}, \{x, z\}$ yields a new graph with the same degrees. With some basic graph techniques for connectivity one can also easily check that the resulting multigraph has underlying graph a forest and has one less component. Note that when $s = r$ the switch produces a new forest with the same number of components.

Since we are working with degrees $d_1/g, d_2/g, \ldots, d_n/g$ with greatest common divisor 1, either there are two edges with different multiplicity and hence two edges in different components with different multiplicity or every edge multiplicity is 1. In the second case, if the multigraph is not connected there are less than $n-1$ edges and the adjusted degree sum is less than $2n-2$. Otherwise we repeat the switching until a connected multiforest, i.e., a multitree, realization is obtained. We note also that the switch can be done so that a given bipartition is preserved.

Observe that here we have used the assumption that we are working with positive degrees rather than nonnegative degrees as we did with multiforests.

We have just outlined a proof of the following.

Fact 15 *Positive integers $d_1, \ldots, d_n$ are the degree sequence of a multitree if and only if they partition into two parts with equal sums and $\frac{\sum_{i\in[n]} d_i}{\gcd(d_1,\ldots,d_n)} \geq 2n-2$. For every partition of the d_i into two parts with equal sum there is a realization for which the degrees in the parts correspond to the partition.*

With similar switching ideas (albeit multiple cases) we can cover two additional situations. For multigraphs we can get connected realizations with only one cycle (of size 3) when the 'reduced' degree sum is large enough. For both multitree and multigraph degrees when the reduced degree sum is small but the greatest common divisor is greater than 1 we can get connected realizations with one cycle which will have a prescribed length. Observe that since these graphs are no longer bipartite, the reduced sum of degrees can be odd. We omit outlining switching ideas for a proof of the following. They are straightforward but involve several cases.

Fact 16 *A sequence of positive integers that is the degree sequence of a multigraph (i.e., $d_1 \geq d_2 \geq \cdots \geq d_n$ with even sum and with $d_1 \leq d_2 + \cdots + d_n$) which also has $\frac{\sum_{i\in[n]} d_i}{\gcd(d_1,\ldots,d_n)} \geq 2n-2$ is the degree sequence of a multitree or the degree sequence of a connected multigraph for which the underlying graph is unicyclic with the cycle having size 3.*

If $\gcd(d_1,\ldots,d_n) > 1$ and $\frac{\sum_{i\in[n]} d_i}{\gcd(d_1,\ldots,d_n)} = 2n-t$ for $t \geq 3$ there is a realization in which the underlying graph is unicyclic with the cycle having size t.

Here we have focused on connected realizations of multigraph sequences with underlying tree or unicylic graph, hence n or $n-1$ underlying edges. The problem of minimizing the number of underlying edges in a multigraph realization is NP-hard [1]. In the other direction, moving away from tree like underlying graphs, maximizing the number of underlying edges corresponds to determining the minimum number of 2's to append to the sequence so that there is a realization as a simple graph [2].

7 Alternate Proofs

We mention three other approaches to prove Fact 8 as well as an alternate proof for Fact 15. For the three alternate approaches to prove Fact 8, the first will use use ideas similar to switching, the second an alternate induction and the third will use Fact 10 and switching. For each of these we provide only a brief sketch and assume some additional basic graph theory terminology and elementary facts about switches, connectivity and cycles. The alternate proof for Fact 15 follows the induction proof of Fact 10 with a little extra work to ensure connectivity.

We will use the following generalized version of edge switching. We say that edge multiplicity 0 corresponds to a non-edge. Assume the pairs in the following are distinct while vertices may repeat. If we increase the multiplicities of
$\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2t-1}, v_{2t}\}$ by m and decrease multiplicities of
$\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{2t}, v_1\}$ by $m \geq 1$ where m is the minimum multiplicity of the second set of edges, the degrees do not change. Here we want $m \geq 1$ while the first set may have pairs with multiplicity 0. The change in the number of underlying edges is the number of 0 multiplicity edges in the first set minus the number of multiplicity m edges in the second set. In particular, if all multiplicities are positive the number of underlying edges decreases and if at most one edge has multiplicity 0 the number of underlying edges does not increase.

In the following 'switch' refers to the generalized version in the previous paragraph.

Proof 2 of Fact 8: This proof uses switching. Given degrees of a multigraph consider a multigraph realization that minimizes the number of edges in the underlying graph. The underlying graph has no even cycles as otherwise a switch decreases the number of underlying edges. A graph with no even cycles has every block an odd cycle or an edge. If there are two odd cycles sharing a vertex one can find an even trail (vertices may repeat but not edges) and switch to reduce the number of edges. So we may assume that the blocks are edges or odd cycles and no pair of odd cycle blocks share a vertex.

If there are two disjoint odd cycles, using two non-edges between these cycles (if they are in different components) or a non-edge and a path between the cycles (if the are in the same component), one can use a switch to form an even cycle then switch again to either obtain fewer cycles or two odd cycles joined by an edge from which using this edge as the path will produce an even cycle on which switching reduces the number of cycles. Repeating results in a graph with at most one odd cycle. If the cycle has 5 or more vertices, switch on a non edge between 2 vertices at distance two and the rest of the cycle to obtain exactly one 3-cycle.

Proof 3 of Fact 8: This proof is a different proof by induction. Given degrees $d_1, d_2, \dots d_n$ of a multigraph replace d_i and d_j with $d_i + d_j$ for any i, j such that $d_i + d_j$ is at most the sum of the other terms. This is possible for $n \geq 4$. When $n = 3$ the basis is the same as described just before Fact 8. The largest degree in the new sequence, whether it is $d_i + d_j$ or d_1 is at most the sum of the other degrees. By induction realize with underlying graph a forest or a unicyclic graph for which the cycle has size 3. 'Split' a vertex of degree $d_i + d_j$ into two vertices. The edges to such a vertex can be partitioned so that only one underlying edge is split to have edges to both new vertices. If the vertex is not on a 3-cycle, any incident edge can be used if an edge must be split. If the vertex is on a 3-cycle then one of the edges of the cycle must be used. In this way the new graph is either still a forest or has at most one 3 cycle.

Proof 4 of Fact 8: This proof builds on the fact that if the degrees partition into two parts with equal sums there is a forest realization. Partition the degrees into two parts such that the difference of sums is as small as possible. As the degree sum is even, the difference between the sums is even. Each value in the larger part is at least the difference in the sums else we can move it to the other part and get a smaller difference. Reduce two values in the larger part by half the difference between the sums to get a new sequence that partitions into two parts with equal sum. Realize the new sequence with a multiforest as in Fact 10. Add multiedges between vertices corresponding to the reduced degrees to get the original degrees realized with exactly one cycle. The cycle is odd since the vertices are in the same part. Finish as in Proof 2. I.e., if the cycle has 5 or more vertices, switch on a non edge between 2 vertices at distance two and the rest of the cycle to obtain exactly one 3-cycle.

Note that in Proof 4 finding a partition into two parts with difference of sums as small as possible is NP-hard, so implementing the proof as a recursive algorithm as stated is not efficient. However, all we really need is two values in the larger part that are at least half the difference between the sums. Finding a minimal difference by repeatedly shifting values less than half the difference from the larger part to the smaller part suffices.

Proof 2 of Fact 15: The basis when $n \leq 3$ is easy to check. Consider sequences $r_1, \dots, r_a$ and $s_1, \dots, s_b$ with $r_1 + \cdots + r_a = s_1 + \cdots + s_b$, the greatest common divisor of all terms equal to 1 and with sum at least $2n-2$ where $a+b=n$. If two values are equal pick so that $r_1 \neq s_b$ and one of r_1 or s_b is equal to some other value in the sequences and in addition pick so that this is the smallest duplicated value. By relabelling we may assume $s_b < r_1$. Consider the sequences $r_1 - s_b, r_2, \dots, r_a$ and $s_1, \dots, s_{b-1}$. It is easy to see that if two values are equal, so that one of s_b or r_1 equals a value in the new sequences then the greatest common divisor of the terms in the new sequences is 1. If the new sum is at most $2(n-1)-3=2n-5$ then at least three new values are 1, hence two original values are 1 and we would have picked $s_b = 1$, contradicting the original sum at least $2n-2$. So when two values are the same the new reduced sum is large enough. If all values are distinct, the new greatest common divisor is g and the new sum divided by g is at most $2(n-1)-3$ then at least 3 of the new values are 1 and hence at least two original values are the same, a contradiction. So when the values are distinct the new reduced sum is large enough. Thus we can get a connected realization for the new values and add add s_b edges between a new vertex and a vertex with degree $r_1 - s_b$ to get a multitree realization.

8 Conclusion

Combining ideas from elementary results on degrees of trees and multigraphs we encounter some additional interesting basic facts on degrees of multigraphs with underlying tree like structure. Another way to combine these basic ideas is to consider degree sequences of 2-multigraphs (where each edge multiplicity is 1 or 2) with underlying graph a tree or forest. In this situation one can also find nice characterizations. These will be presented in another paper and can be left as an exercise for the reader here. For now we note that results regarding partitions do not carry over to 2-multitrees. Our results noted that for any partition of a sequence into two parts with equal sum there is a multiforest or multitree realization for which the degrees in the parts correspond to the given partition of the sequence. For the case of 2-multigraphs with underlying graph a tree or forest the partition matters. There are examples where some equal sum partitions give a 2-multitree realization and others do not.

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