Chain Packings and Odd Subtree Packings

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Abstract

A chain packing H in a graph is a subgraph satisfying given degree constraints at the vertices. Its size is the number of odd degree vertices in the subgraph. An odd subtree packing is a chain packing which is a forest in which all non-isolated vertices have odd degree in the forest. We show that for a given graph and degree constraints, the size of a maximum chain packing and a maximum odd subtree packing are the same but the same does not hold for a version in which the sum of given weights on the odd degree vertices is to be maximized. We also note a reduction to weighted capacitated b-matching for finding a maximum size chain packing, maximum size odd subtree packing and maximum weight chain packing. The main result of this note is the proof that a min-max formula generalizing the Berge-Tutte formula for matching holds for chain packing.

1 Introduction

Edge disjoint packings of chains as an extension of matching have been studied by deWerra [14, 15, 16], deWerra and Pulleyblank [13] and deWerra and Roberts [17]. In [14, 15, 16, 13] the chains have odd length. Chain Packings of general length are studied in [17]. A *chain packing* H in a graph G is a subgraph of G satisfying given degree constraints. Its *size* is the number of vertices with odd degree in H. We will examine chain packings and a closely related notion of odd subtree packings. An *odd subtree packing* F is a chain packing for which each non-trivial component is a tree containing no even degree vertices. Its *size* is the number of odd degree vertices in F. Thus, odd subtree packing can be viewed as a generalization of matching in which matched edges are replaced by odd subtrees satisfying certain degree constraints. Related, but distinct problems of packing with the subgraphs drawn from a fixed family and finding subgraphs satisfying degree constraints have been extensively studied. See for example [1], [7] and [9] and the references therein.

The maximum size of a chain packing and the maximum size of an odd subtree packing are easily shown to be equal. However, for a given graph, the collections of sets of vertices which have odd degree in a chain packing and in an odd forest packing are different. In particular, when weights are assigned to the vertices and we wish to maximize the sum of the weights of the odd vertices, the maximum value for chain packings can be larger than the maximum value for odd forest packings.

We will examine the similarities and differences of chain packings and odd forest packings and note a reduction to weighted capacitated b-matching. The main result of this note is the proof that a min-max formula generalizing the Tutte-Berge formula for matching holds for the maximum size of a chain packing. We show that the maximum size of a chain packing (for which all degree constraints are odd) is equal to the minimum over all subsets S of the vertex set of

$$|V| + b(S) - \mathcal{O}(G \setminus S)$$

where $b(S)$ is the sum of the constraints over vertices in S. This answers a question posed in deWerra and Roberts [17].

We briefly mention the graph theoretic notation which we will use. See for example [4] for any terms not defined here. We will assume that all graphs $G = (V, E)$ are finite and contain no loops unless otherwise noted. Vertex x is *adjacent* to vertex y if $\{x, y\}$ is an edge in $E(G)$. A *chain* in a graph is a sequence $v_0, v_1, \ldots, v_k$ such that $\{v_i, v_{i+1}\}$ is an edge for $i = 0, \ldots, k-1$. If the vertices are distinct we call this a *path*. A graph is *connected* if there is a chain between every pair of vertices. A *component* of a graph is a maximal connected subgraph. A *matching* is a subgraph for which each component is an edge. The *degree* of a vertex v in a graph $G = (V, E)$, denoted by $d_G(v)$ is the number of edges in $E(G)$ with one end v. A vertex is *isolated* if it has degree 0. A *forest* is an acyclic digraph and it is called a *tree* if it is connected. We will use $G \setminus S$ to denote the graph induced by the vertices of $V(G) \setminus S$ if S. We will also use $G \setminus \{\{x, y\}\}$ to denote the graph G with edge $\{x, y\}$ removed.

2 The Problems

In this section we give formal definitions of the problems we will consider and review the relationships between these problems. In both weighted and unweighted versions

Chain Packing: Given a graph $G = (V, E)$ and positive integer constraints $b : V \to \mathbf{Z}^+$, a *chain packing* in G is a subgraph $H = (V, P)$ that satisfies $d_H(v) \leq b(v)$ for all $v \in V$. The *size* of a chain packing is the number of vertices with odd degree in H. If weights w are assigned to the vertices, we call the problem *weighted chain packing* and the weight of a packing is the sum of the weights of the vertices which have odd degree in H.

As noted in deWerra and Roberts [17], the chain packing subgraph H can be decomposed into a collection of edge disjoint chains (plus possibly some cycles) with the chains having distinct endpoints. Thus the name chain packing. The size of the chain packing is twice the number of chains in the decomposition, with each odd degree vertex in H appearing as the endpoint of one chain. To find such a decomposition, we simply find a chain C in H connecting two odd degree vertices. Add this to the collection of chains and continue the decomposition with $H \setminus C$ until the remaining graph has only even degree vertices. A graph with all even degrees can be decomposed into a collection of cycles.

Also, deWerra and Roberts [17] note that when looking for a chain packing of maximum size, one may assume that the constraints are all odd. Let H be a chain packing of maximum size. If $d_H(v) = b(v) > 0$ for some even $b(v)$, then consider $H' = H \setminus \{\{v, w\}\}$ for any edge $\{v, w\} \in H$. We have $d_{H'}(v) = d_H(v) - 1$, $d_{H'}(w) = d_H(w) - 1$, and $d_{H'}(x) = d_H(x) - 1$ for $x \in V$, $x \neq v, w$. Thus H' satisfies the degree constraints. If $d_H(w)$ is even then H does not have maximum size. So, in H, v has even degree and w

has odd degree, and in H', v has odd degree and w has even degree. H' is also a maximum size chain packing. By continuing such reductions, we can find a maximum size chain packing H for which there are no vertices with $d_H = b(v)$ and $b(v)$ even. Thus, deWerra and Roberts [17] assume that all degree constraints are odd. Note that the reductions described above do change the set of odd degree vertices. We will discuss the differences between chain packings in a graph and chain packings in the same graph when all even degree constraints are reduced by one. Thus, we will distinguish between chain packings with general constraints and chain packings for which the constraints are all odd, even though in general the second is a special case of the first.

Odd Chain Packing: A chain packing in which all degree constraints are odd is called an *odd chain packing*. In particular an instance of odd chain packing obtained from an instance of chain packing is the odd chain packing problem obtained by setting $b(v) = b(v) - 1$ for all even $b(v)$. If weights w are assigned to the vertices we call the problem *weighted odd chain packing* and the weight of a packing is the sum of the weights of the vertices which have odd degree in H.

Given a set of constraints b on a set V we will define the the *reduced* constraints $\tilde{b}$ by $\tilde{b}(v) = b(v)$ if $b(v)$ is even and $\tilde{b}(v) = b(v) - 1$ if $b(v)$ is odd.

Consider a maximum size odd chain packing H. Removing a cycle from H does not change the parity or increase the degree of any vertex. Thus, there is a maximum size odd chain packing which is a forest. Furthermore, consider a forest F containing a non-isolated vertex w with even degree. We can find a path in F from w to a vertex x with $d_F(x) = 1$ (a leaf in the forest F). By deleting such a path, we obtain a new forest F' in which x is isolated and $d_{F'}(w) = d_F(w) - 1$. So w has odd degree. Also, the degree of every other vertex is unchanged or reduced by 2. Removing paths from non-isolated vertices of even degree produces a forest F' with the same number of odd degree vertices as F and $d_{F'}(v) \leq d_F(v)$ for all $v \in V$. Thus, a maximum size chain packing and a maximum size odd chain packing can always be realized by a forest in which all the non-isolated vertices have odd degree. However, the reductions to an odd forest change the set of odd degree vertices. Thus we define a third problem, that of odd subtree packing.

Odd Subtree Packing: Given a graph $G = (V, E)$ and odd positive integer constraints $b : V \to 2\mathbf{N} + 1$, an *odd subtree packing* $F = (V, P)$ is a subgraph

G which is a forest such that for all $v \in V$, either $d_F(v) = 0$ or $d_F(v)$ is odd. Furthermore, $d_F(v) \leq b(v)$ for all $v \in V$. The **size** of an odd forest is the number of vertices with odd degree. If weights w are assigned to the vertices we call the problem *weighted odd subtree packing* and the weight of a packing is the sum of the weights of the vertices which have odd degree in F.

Note that for each of the problems chain packing, odd chain packing and odd subtree packing, maximum size matching arises as a special case when $b(v) = 1$ for all $v \in V$. The maximum size of a chain packing (odd subtree packing) in this case is twice the size of a maximum matching. However, maximum weight matching is not a special case, as we consider weights on the vertices, not the edges.

We have already discussed the similarities between the unweighted versions of the three problems. We give examples below to show that the weighted versions of these problems may all have different maximums. Thus we have the following.

Remark 1 *Let a graph $G = (V, E)$ and positive integer constraints b on V be given. Let $\tilde{b}$ denote the reduced constraints obtained from b. The maximum size of a chain packing on G with constraints b, the maximum size of an odd chain packing on G with constraints $\tilde{b}$, and the maximum size of an odd subtree packing on G with constraints $\tilde{b}$ are all equal. However, the collections of sets of vertices which have odd degree in some maximum chain packing, maximum odd chain packing, and maximum odd subtree packing may all be different. Additionally, given weights on the vertices, the weighted versions of the problems may all have different maximums.*

We show in Figure **??** an example demonstrating the last comment in the remark. Let $b(v) = d_G(v)$ for every vertex in the graph G in the figure. Then $b(v) = 2$, so $\tilde{b}(v) = b(v) - 1 = 1$, $b(u) = \tilde{b}(u) = b(x) = \tilde{b}(x) = b(y) = \tilde{b}(y) = 1$, and $b(w) = \tilde{b}(w) = 3$. It can can be checked that the maximum size chain packings on G with constraints b are G, $G_1 = G \setminus \{\{w, v\}\}$ (G with edge $\{w, v\}$ deleted), $G_2 = G \setminus \{\{u, v\}\}$ (G with edge $\{u, v\}$ deleted), $G_3 = G \setminus \{\{v, w\}, \{w, y\}\}$, and $G_4 = G \setminus \{\{v, w\}, \{w, x\}\}$. The maximum size odd chain packings on G with constraints $\tilde{b}$ are G_1, G_2, G_3, and G_4. The maximum size odd subtree packings on G with constraints $\tilde{b}$ are G_2, G_3, and G_4. Thus, the corresponding collections of sets of vertices covered by the maximum packings are all distinct. Assign the weights $\tilde{w}(u) = 3$, $\tilde{w}(v) = 1$, $\tilde{w}(w) = 2$, $\tilde{w}(x) = \tilde{w}(y) = 4$ to the vertices of G. Then a maximum weight chain packing with respect to b has weight 13, a maximum weight odd chain packing with respect to $\tilde{b}$ has weight 12, and a maximum weight odd subtree packing with respect to $\tilde{b}$ has weight 11.

Call a set S of vertices *saturable* with respect to a given type of packing if there exists a packing of that type in which all vertices in S have odd degree. In dewerra and Roberts [17], the saturable sets in a graph with respect to odd chain packing are shown to be a matroid. By modifying their definition of augmenting chain to allow repeated end appearances of end vertices, their proof can be seen to show that the saturable sets in a graph with respect to general chain packing also form a matroid. The algorithm described in Isaak [8] shows that the saturable sets in a graph with respect to odd subtree packing also form a matroid.

3 Algorithms

In this section we present a reduction to weighted capacitated b-matching which shows that there is a polynomial algorithm to find the maximum size or maximum weight of a chain packing. Thus, the reduction also can be used to give a maximum size odd subtree packing. However, the reduction does not work for maximum weight odd subtree packing. An alternative approach (details of which can be found in [8]) that directly computes the maximum size of a chain packing or odd subtree packing is also briefly discussed.

We first recall a formulation of the problem of weighted capacitated b-matching. See for example [6] or [3] for more details on b-matching. Given a graph $G = (V, E)$ with 'loops' $x_l(v)$ for each vertex $v \in V$, capacities $b(v)$ for each vertex, and capacities c_e and weights w_e on the edges (including the loops), a weighted capacitated b-matching is an assignment of non-negative integers x_e to the edges (including the loops) such that $x_e \leq c_e$ and for each $v \in V$, $2x_l(v) + \sum x_e \leq b(v)$ where the sum is over all edges with one end v. A perfect b-matching is one in which $2x_l(v) + \sum x_e = b(v)$ for all $v \in V$. The weight of the b-matching is $\sum w_e x_e$ where the sum is over all edges. There are known polynomial procedures for finding a minimum weight b-matching. See for example Anstee [3] and the references there.

We will first describe a construction reducing maximum size chain packing to weighted capacitated perfect b-matching. We will then show how this can be easily modified for weighted chain packing. This reduction is due to an anonymous referee.

The construction reducing maximum size chain packing to weighted capacitated perfect b-matching is similar to a construction used in Edmonds and Johnson [6]. Let a graph $G = (V, E)$ and positive integer constraints

$b(v)$ on the vertices be given. Let $EVEN$ be the set of vertices with even constraints and ODD the set of vertices with odd constraints. Construct a new graph G' with weights and capacities on the edges as follows. Let each edge in $E(G)$ have weight 0 and capacity 1. Add a new vertex z and new edges $\{z, v\}$ for all $v \in V$. Let $b(z)$ be either $|V|$ or $|V| - 1$, so that the parity of $b(z)$ agrees with the parity of $|ODD|$ (i.e., the number of vertices with odd degree constraints). This insures that G' will have a perfect b-matching. Let the new edges have capacity 1. Set the weight on the new edge $\{v, z\}$ to 1 if $v \in ODD$ and -1 if $v \in EVEN$. Finally, for each vertex (including the new vertex z) add a loop with weight 0 and 'large' capacity (at least $b(v)/2$). Then, it can be seen that a minimum weight capacitated perfect b-matching in G' corresponds to a maximum size chain packing in G and vice versa.

If B is the set of edges (with multiplicities) in a b-matching in G', then edges of G appear in B with multiplicity at most one due to the capacity constraints. Let $P = B \cap E(G)$, the edges of B in G. The edges of P are the edges of a packing, since the degree constraints are the same in the b-matching and the chain packing. Also, in the graph induced by P, a vertex $v \in EVEN$ has even degree if $\{v, z\} \notin B$ and odd degree if $\{v, z\} \in B$. A vertex $v \in ODD$ has even degree if $\{v, z\} \in B$ and odd degree if $\{v, z\} \notin B$. Given a packing P, a b-matching can be formed by adding edges adjacent to z by the conditions of the preceding sentence. Then the parity of each vertex is the same as its constraint and the b-matching can be completed by adding loops with appropriate multiplicity at each vertex (including z). So there is a one-to-one correspondence between b-matchings in the new graph and chain packings in the original graph.

Consider a chain packing P and the corresponding b-matching B. Let $EVEN^-$ and $EVEN^+$ denote, respectively, the vertices in $EVEN$ with odd degree and with even degree. Similarly, let ODD^- and ODD^+ denote the vertices in ODD with odd and respectively even degree in P. Then, $|ODD| = |ODD^-| + |ODD^+|$ and the size s of the packing is given by $s = |ODD^-| + |EVEN^-|$. If t is the weight of the b-matching, we have (from the description of edges incident to the new vertex z),

$$t = |ODD^+| - |EVEN^-| = |ODD^-| + |ODD^+| - \big(|ODD^-| + |EVEN^-|\big) = |ODD| - s.$$

Thus, since $|ODD|$ is constant, maximum size chain packings correspond to minimum weight b-matchings.

For weighted chain packings, if $\tilde{w}$ are the weights on the vertices of G, then we construct G' as above, except that we set the weight of the edges

$\{v, z\}$ to $\tilde{w}(v)$ if $v \in ODD$ and $-\tilde{w}(v)$ if $v \in EVEN$. The weight of the packing s is $s = \sum_{v \in (ODD^-)} \tilde{w}(v) + \sum_{v \in (EVEN^-)} \tilde{w}(v)$. Then in a manner analogous to unweighted chain packing, for the weight t of the b-matching, we get $t = \sum_{v \in ODD} \tilde{w}(v) - s$. So the reduction can also be used to determine maximum weight chain packings using b-matching.

Note that using the reductions described in Section 2, a maximum size chain packing can be used to construct a maximum size odd subtree packing. So the b-matching reduction can also solve the problem of finding a maximum size odd subtree packing. We can obtain a maximum size odd subtree packing directly from the minimum weight b-matching by putting weights of $1 >> \epsilon > 0$ on the edges of G in G'. It can be seen that a minimum weight b-matching in G' will correspond to a maximum size odd subtree packing (in a fact a maximum subtree packing with a minimum number of edges). However, we note that this method does not seem to work to solve the maximum weight odd subtree packing problem.

Remark 2 There exist polynomial procedures to find maximum weight and maximum size chain packings and maximum size odd subtree packings.

Finally, we briefly comment that using an augmenting chain theorem of deWerra and Roberts [17] provides a more direct procedure for maximum size odd chain packing and maximum size odd subtree packing. This procedure is a blossom type algorithm along the lines of that of Edmonds [5], except that more general conditions for blossoming are used. The complexity is $O(|V|^3)$. See Isaak [8] for details. We note also that the augmenting chain theorem of deWerra and Roberts [17] does not necessarily hold for the general case of chain packing (with even degree constraints). Their definition does not allow an end vertex of an augmenting chain to have multiple appearances in the chain. In order for their augmenting chain theorem to work in the general case, the definition of augmenting chains must be slightly modified to allow end vertices to have repeated appearances. See deWerra and Roberts [17] for details on this augmenting chain theorem.

4 A Min-Max Formula

We have already observed that for a given graph $G = (V, E)$, constraints b, and the reduced constraints $\tilde{b}$ obtained from b, that the maximum size of a chain packing in G with constraints b, the maximum size of an odd chain packing in G with constraints $\tilde{b}$ and the maximum size of an odd subtree packing in G with constraints $\tilde{b}$ are all equal. In this section we state a min-max formula for this common value which is a straightforward generalization of the Tutte-Berge formula for the maximum size of a matching. For simplicity, we will consider the problem of odd chain packing.

We will first prove a special case of the min-max formula for chain packings in which all vertices have odd degree.

Definition 1 A *perfect* odd chain packing on G is an odd chain packing with size $|V(G)|$.

Let $\mathcal{O}(G)$ denote the number of odd components in the graph G. Let $b(S) = \sum_{v \in S} b(v)$ for any set of vertices $S \subseteq V$.

The Tutte-Berge formula for maximum cardinality matching states that the size of a maximum cardinality matching is equal to

$$\min_{S \subseteq V} \frac{|V| - (\mathcal{O}(G \setminus S) - |S|)}{2}.$$

That is, the number of vertices left uncovered by the matching is equal to the maximum value of $\mathcal{O}(S) - |S|$ over all subsets S of the vertex set V. Numerous proofs of this result can be found. See for example Lovász and Plummer [11] or Bollabás [4]. We will prove that an analogous formula for odd chain packing holds.

In chain packing, as in matching, half of the min-max formula is immediate. Let a graph G with odd positive integer constraints b be given. Consider any set $S \subseteq V(G)$. At most $b(S)$ edges can have exactly one end in S in any chain packing P. Let C_i be an odd component of $G \setminus S$. By parity, either some vertex of C_i is even in P, or there is an edge of P with exactly one end in C_i. Since C_i is a component of $G \setminus S$, an edge with exactly one end in C_i has the other end in S. Thus, of the $\mathcal{O}(G \setminus S)$ odd components of $G \setminus S$, at most $b(S)$ can have all vertices odd in P. That is, for any S, at least $\mathcal{O}(G \setminus S) - b(S)$ vertices must be even in any packing P. This shows that the maximum size of a chain packing is less than or equal to

$$\min_{S \subseteq V} |V| - (\mathcal{O}(G \setminus S) - b(S)).$$

We will show that equality does hold.

A special case of the Tutte-Berge formula for matching is that a graph has a perfect matching if and only if

$$\mathcal{O}(S) \leq |S| \tag{1}$$

holds for all subsets S of the vertex set V. For odd chain packing, we will show that there is a perfect odd chain packing in G if and only if

$$\mathcal{O}(S) \leq b(S) \qquad (2)$$

holds for all subsets S of the vertex set V. As in the case of matching, the condition for perfect odd chain packings will be used to provide a general min-max formula.

We give a proof which follows very closely the proof of the Tutte-Berge formula for matching given by Mader [12] and Anderson [2] as presented in Bollabás [4]. There are a few minor differences in our proof which are not needed in the proofs of Mader and Anderson for the matching result. In both cases, note that if (1) or (2) holds, then $|V(G)|$ must be even. For matching, if S is a maximal set with (1) holding with equality, then $S \neq \emptyset$ (as equality holds for single element sets) and $G \setminus S$ contains no even components (as equality will also hold for $S \cup \{x\}$ for any x in an even component). In odd forest packing, similar facts do not hold. If (2) holds in G, and if S is maximal with equality in (2), then S may be empty and also $G \setminus S$ may contain non-trivial even components (as $\mathcal{O}(G \setminus (S \cup \{x\}))$ may be only $\mathcal{O}(G \setminus S) + 1$ while $b(S \cup \{x\})$ may be strictly greater than $b(S) + 1$ if $b(x) > 1$). For example, if G is the complete graph on six vertices and all degree constraints are three, then the only set for which equality holds in (2) is $\emptyset$ and $G \setminus \emptyset$ has an even component.

Assume that (1) or (2) holds in G. A second difference in the case of odd forest packing occurs in the odd components of $G \setminus S$ for S maximal such that equality holds in (1) or (2). In matching, if S is maximal such that equality holds in (1) then the proof of Mader and Anderson shows that if C is an odd component of $G \setminus S$ then C is factor critical, i.e., $C \setminus \{x\}$ has a perfect matching for every $x \in V(C)$. In the case of odd forest packing, if S is maximal such that equality holds in (2), and if C is an odd component of $G \setminus S$, it is not necessarily the case that $C \setminus \{x\}$ has a perfect odd forest packing for every $x \in V(C)$. For example, with the star consisting of edges $\{x, u\}, \{x, v\}, \{x, w\}$ if we have $b(u) = b(v) = b(w) = 1$ and $b(x) = 3$, then $\{u\}$ is maximal such that equality holds in (2). There is one odd component of $G \setminus \{u\}$, the graph C consisting of the two edges $\{x, v\}, \{x, w\}$. Deleting x from C produces a graph with two isolated vertices, i.e., a graph with no perfect odd chain packing. However, if we add a new pendant edge from x in C we get a new graph isomorphic to G, which does have a perfect odd chain packing. This example suggests considering the following graphs.

Definition 2 Let a graph G and degree constraints b be given. For each

$x \in V(G)$, let G_x be formed by adding to G a new vertex z with $b(z) = 1$ and a pendant edge $\{x, z\}$ joining x and z. G is *extension perfect* if G_x has a perfect odd subtree packing for each x in $V(G)$.

Note that if $b(v) = 1$ for all $v \in V(G)$ and if G is factor critical, then G is extension perfect. If S is maximal such that equality holds in (2) and if C is an odd component of $G \setminus S$, then we will show that C is extension perfect.

With these slight variations in mind we can show that the analog of the Tutte-Berge formula holds for odd forest packing. We first prove the result for perfect odd forest packings. Note that there is no loss of generality in assuming that $b(v) \leq d_G(v)$ for all vertices v.

Theorem 1 *Let $G = (V, E)$ be a graph and let $b : V \to \mathbf{2N+1}$ be odd degree constraints (such that $b(v) \leq d_G(v)$ for all v). G has a perfect odd chain packing if and only if*

$$b(S) \geq \mathcal{O}(G \setminus S) \text{ for all } S \subseteq V. \tag{3}$$

Proof: As noted previously, if $b(S) < \mathcal{O}(G \setminus S)$ for some S, then in any chain packing, some odd component of $G \setminus S$ must contain a vertex of even degree in the packing, so G does not have a perfect odd subtree packing.

We prove the converse by induction on $|V(G)| + |E(G)| + \sum b_i$. Assume that G is such that (3) holds. Note that $|V(G)|$ is even or else (3) is violated by $S = \emptyset$. Since the constraints b are all odd, the parity of $b(S)$ is the same as the parity of S. If $|V(G)|$ is even, then by parity

$$b(S) > \mathcal{O}(G \setminus S) \Longrightarrow b(S) \geq \mathcal{O}(G \setminus S) + 2. \tag{4}$$

If $|V(G)| = 2$, G is either an edge or two isolated vertices and the result can easily be seen to hold. So $|V(G)| \geq 4$.

We may assume that G is connected or else by induction on each component we are done.

Let S be a maximal set such that equality holds in (3). Assume first that $S = \emptyset$. There can be no vertex w such that $b(w) = 1$ since equality would hold in (3) for $S = \{w\}$ contradicting the maximality of $\emptyset$. Pick any vertex w and change $b(w)$ to $b(w) - 2$. By (4), since $>$ holds in (3) under the old constraints for $S \neq \emptyset$, equation (3) still holds under the new constraints. So, by induction, the graph with the reduced constraints (and thus the original graph) has a perfect odd chain packing.

Next, assume that $b(S) = 1$ for all maximal sets for which equality holds in (3). Since the $b(v)$ are positive, the maximal sets also must satisfy $|S| = 1$. If $b(v) > \mathcal{O}(G \setminus \{v\})$ for some vertex v. Then, by the strict inequality for $\{v\}$

and since all maximal sets contain one element, we have $b(S) > \mathcal{O}(G \setminus S)$ for all $S \subseteq V(G)$ with $v \in S$. As above we can change $b(v)$ to $b(v) - 2$, and we are done by induction.

Assume now that $b(S) = 1$ for all maximal set S for which equality holds in (3) and that all single element sets are maximal. Additionally, assume that for all $v \in V(G)$, $G \setminus \{v\}$ has no even components. (Note that we have $b(v) = 1$ for all $v \in V(G)$ and we could use the Tutte-Berge formula to complete the proof of this case. However, we will give a short alternative proof so that we do not need to rely on the matching result directly.) Then, if there exists a vertex x such that $b(x) < d_G(x)$, consider the graph $G \setminus \{x\}$. Since $G \setminus \{x\}$ has no even components and since $d_G(x) > b(x)$ and $b(x) = \mathcal{O}(G \setminus \{x\})$ ($\{x\}$ is maximal), there is one (odd) component C in $G \setminus \{x\}$, and x is adjacent to at least two vertices in C. Let y and z be two such vertices. Since y is adjacent to x, $d_G(y) \geq 2$ or else y and z are not in the same component of $G \setminus \{x\}$. Now, in $G' = G \setminus \{\{x, y\}\}$ (G with edge $\{x, y\}$ deleted), $b(v) \leq d_{G'}(v)$ holds for all vertices (by the choice of $\{x, y\}$). Additionally, it is not difficult to check that since (3) holds in G, the only way for (3) to be violated in G' is if $b(S) = \mathcal{O}(G \setminus S)$ and removing the edge $\{x, y\}$ from some even component of $G \setminus S$ produces two odd components. However, we have assumed that the only sets for which equality in (3) holds in G are such that $G \setminus S$ contains no even components. Thus, (3) holds in G' and this case is complete by induction.

Under the same assumptions as the preceding paragraph, if $d_G(x) = b(x) = 1$ for all $x \in V$, then G is a collection of edges and G itself is a perfect odd chain packing.

Finally, we consider the cases that there is a maximal $|S|$ with $b(S) \geq 2$ or there is a maximal S with $b(S) = 1$ and some component of $G \setminus S$ even. Let $k = b(S)$. Let $D_1, \ldots$ be the even components and $C_1, \ldots, C_k$ the odd components of $G \setminus S$. If some $S' \subseteq V(D_i)$ violates (3) in D_i, then $S \cup S'$ violates (3) in G. So (3) holds in each even component D_i and by induction each of these components has a perfect odd chain packing.

We next show that every odd component of $G \setminus S$ is extension perfect. Consider any such component $C = C_i$ and any $y \in V(C)$. Construct C' with $V(C') = V(C) \cup \{z\}$ (for a new vertex $z \notin V(G)$) and $E(C') = E(C) \cup \{\{z, y\}\}$. That is, C' is formed from C by adding a new pendant edge $\{y, z\}$ to y. Let $b(z) = 1$. By assumption $|V(C')| + |E(C')| + \sum_{v \in V(C')} b_v < |V(G)| + |E(G)| + \sum_{v \in V(G)} b_v$ since either $b(S) \geq 2$ or there is an even component D_1 with at least two vertices. We show that C' satisfies (3). Assume that there is a set $S' \subseteq V(C')$ which violates (3) in C'. Clearly

$S' \neq \emptyset$ since C' is connected and $|V(C)|$ is even. Also, it is not difficult to check that if $S' \cup \{z\}$ violates (3) in C', then S' violates (3). So we assume that $z \notin S'$. Note that $\mathcal{O}(C \setminus S') \geq \mathcal{O}(C' \setminus S') - 1$ since at most one odd component becomes even (or empty) by removing the pendant vertex z. Since S' violates (3) in C', $b(S') < \mathcal{O}(C' \setminus S')$. Then, by parity, $b(S') \leq \mathcal{O}(C' \setminus S') - 2$. Since S' is contained in the vertex set of some odd component C of $G \setminus S$, we have $\mathcal{O}(G \setminus S \setminus S') = \mathcal{O}(G \setminus S) - 1 + \mathcal{O}(C \setminus S')$. So, we have

$$\begin{aligned} b(S \cup S') &= b(S) + b(S') \\ &\geq \mathcal{O}(G \setminus S \setminus S') \\ &\geq \mathcal{O}(G \setminus S) - 1 + \mathcal{O}(C \setminus S') \\ &\geq \mathcal{O}(G \setminus S) - 1 + \mathcal{O}(C' \setminus S') - 1 \\ &\geq b(S) + b(S') \end{aligned}$$

Thus, in G, equality holds in (3) for $S \cup S'$, contradicting the maximality of S. So, (3) holds in C' and thus by induction, C' has a perfect odd chain packing. That is, C is extension perfect.

To finish the proof, we show that for each odd component C_i we can replace the pendant edge to the new vertex z in C'_i with an edge to a vertex of S, completing the perfect odd chain packing. Form a bipartite graph G' on vertex set $A \cup C$ with A containing $b(v_i)$ vertices $a_{i1}, \ldots, a_{iv_i}$ corresponding to each $v_i \in S$ and each vertex c_i of C corresponding to an odd component C_i of $G \setminus S$. Note that since $b(S) = \mathcal{O}(G \setminus S)$, $|A| = |C|$. Place an edge between $c_i \in C$ and $a_{jl} \in A$ if some vertex of the odd component C_i corresponding to c_i is adjacent to the vertex v_j in S corresponding to a_{jl}. Consider any subset C'' of C, which we may assume to be $c_1, \ldots c_s$. By relabeling, denote the set of vertices of S adjacent to some vertex of some C_i, $i = 1, \ldots, s$ by $S'' = v_1, \ldots, v_t$. By construction, $A'' = \{a_{jk} | 1 \leq j \leq t, 1 \leq k \leq b_{v_j}\}$ is the set of vertices adjacent to C'' in G'. We have $|A''| = b(S'')$ and by (3) in G, we have $b(S'') \geq \mathcal{O}(G \setminus S'')$. Since the adjacencies of the components C_i, $i = 1, \ldots, s$ are all in S'', we have $\mathcal{O}(G \setminus S'') \geq s$. So for any $C'' \subseteq C$,

$$|A''| = b(S'') \geq \mathcal{O}(G \setminus S'') \geq s = |C''|.$$

By Hall's Theorem for matching in bipartite graphs, G' has a perfect matching.

We can use the perfect matching in G', along with the fact that the odd components are extension perfect to construct a perfect matching in G. (We have already noted that the even components of $G \setminus S$ have a perfect matching.) Take any perfect matching M in the bipartite graph G'. The vertex corresponding to an odd component C_i is adjacent to some a_{jl} in M. Thus, in G, some vertex x of C_i is adjacent to $a_j \in S$. Since C_i is extension perfect, there is a perfect matching in C_i with a pendant edge from x. Take this pendant edge to be $\{x, a_j\}$. Doing this for all odd components produces a packing in which all vertices of all odd components are odd. Additionally, from the construction of the bipartite graph G' and from the choice of pendant edges, each vertex $v \in S$ has degree $b(s)$ in the packing, and since $b(S)$ is odd, the vertices of S also have odd degree. □

We can now state as a corollary a general min-max formula

Corollary 2 *Given a graph G and odd positive degree constraints b on $V(G)$, the maximum size of an odd chain packing in G is equal to*

$$\min_{S \subseteq V} |V| - (\mathcal{O}(G \setminus S) - b(S)) = \min_{S \subseteq V} |V| + b(S) - \mathcal{O}(G \setminus S).$$

Proof: We have previously noted that min $\leq$ max holds. To show that equality holds, let m denote the maximum over subsets S of $V(G)$ of $\mathcal{O}(G \setminus S) - b(S)$. So $|V| - m$ is the value of the minimum in the statement of the corollary. Form G' by taking G and a vertex disjoint copy of K_m (the complete graph on m vertices) and joining every vertex of G to every vertex of K_m. Let $b(x) = 1$ for $x \in V(K_m)$. The parity of m and $V(G)$ are the same, so $|V(G')|$ is even. Thus (3) holds for $\emptyset$ in G'. Consider $S \neq \emptyset$, $S \subseteq V(G')$. If S does not contain $V(K_m)$, then $G \setminus S$ is connected and $\mathcal{O}(G' \setminus S)$ is 0 or 1. Since $S \neq \emptyset$, $b(S) \geq 1$ and (3) holds in this case. If S contains $V(K_m)$, then for $S' = S \setminus V(K_m)$, we have $\mathcal{O}(G' \setminus S) = \mathcal{O}(G \setminus S')$. Note that $b(S) = b(S') + m$. Now, $\mathcal{O}(G \setminus S') - b(S') \leq m$ so $\mathcal{O}(G' \setminus S) \leq b(S)$ holds. By the theorem, G' has a perfect odd subtree packing. Such a packing restricted to G is a chain packing in G with $\min_{S \subseteq V} |V| + b(S) - \mathcal{O}(G \setminus S)$ odd vertices. □

By Remark 1, the min-max formula holds for odd subtree packing. Note that the min-max formula for chain packing reduces to the Tutte-Berge formula for matching; since the size of a maximum chain packing when all constraints are one is twice the size of a maximum matching (the matching counts edges and the packing counts odd vertices).

Finally, we note that a direct formula can be obtained when even degree constraints are retained. For this, let $E(S)$ be the number of vertices v in S with $b(v)$ even. Then, in a manner similar to the above proofs, it is not difficult to see that the maximum size of a general chain packing is equal to

$$\min_{S\subseteq V} |V| - E(S) - (\mathcal{O}(G \setminus S) - b(S)) = \min_{S\subseteq V} |V| + b(S) - E(S) - \mathcal{O}(G \setminus S).$$

5 Conclusion

We have seen that a min-max formula holds for chain packing and odd subtree packing. We are also able to show that a decomposition along the lines of that of Edmonds and Gallai for matching (see for example [11]) holds for chain packing (in preparation), but do not have a similar result for odd subtree packing. The reduction to b-matching leaves open the problem of finding an efficient algorithm in the weighted case of odd subtree packing. It would also be natural to consider which other matching 'results' have interesting analogues when extended to chain packing or odd subtree packing.

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