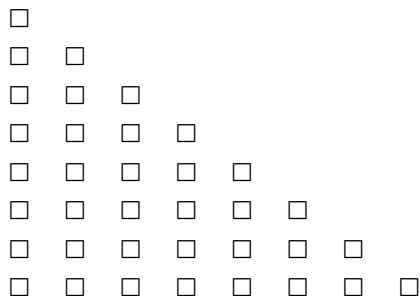


The triangular numbers are the number of items in a triangular stack. We write this as  $T_n$  if there are  $n$  items in the bottom row. For example, in a stack with 8 boxes on the bottom row we have a total of  $T_8 = 1 + 2 + \dots + 8 = 36$  boxes.



Observe that the sequence  $T_1, T_2, T_3, \dots = 1, 3, 6, 10, 15, 21, 36, \dots$  corresponds to the third column of the binomial triangle so we might expect  $T_n = \binom{n+1}{2} = \frac{(n+1)n}{2}$ . This is indeed the case.

There are many elementary proofs of this fact. For example write  $T_n$  twice, once with the numbers in reverse order:

$$T_n = 1 + 2 + 3 + \dots + n$$

$$T_n = n + n - 1 + n - 2 + \dots + 1$$

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$$2T_n = (n + 1) + (n + 1) + (n + 1) + \dots + (n + 1)$$

There are  $n$  terms on the right so  $2T_n = n(n + 1)$  or  $T_n = \frac{n(n+1)}{2}$ .

In order to illustrate induction we will give an proof by induction even though there are much shorter proofs.

The triangular numbers satisfy  $T_n = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$  for  $n = 1, 2, \dots$

Proof: By induction. The formula is trivial when  $n = 1$  as  $T_1 = 1 = \frac{1(1+1)}{2}$ . By induction we may assume that  $1 + 2 + \dots + (n - 1) = \frac{(n-1)(n-1+1)}{2} = \frac{(n-1)n}{2}$ . Then  $T_n = (1 + 2 + \dots + n - 1) + n = \frac{(n-1)n}{2} + n = \frac{(n-1)n}{2} + \frac{2n}{2} = \frac{n((n-1)+2)}{2} = \frac{n(n+1)}{2}$ . So the formula holds for  $n$  and by induction the formula holds for all  $n = 1, 2, \dots$ ,  $\square$

We can consider in general sums of powers of integers  $T_n^k + \sum_{i=1}^n i^k$  for other powers. There are methods to work out  $T_n^k$  in terms of  $T_n^j$  for  $j < k$  but there is no simple general expression for these. We will illustrate this for sums of squares.  $n^3 = \sum_{i=1}^n [i^3 - (i - 1)^3] = \sum_{i=1}^n [i^3 - (i^3 - 3i^2 + 3i - 1)] = \sum_{i=1}^n (3i^2 - 3i + 1)$ . Thus  $n^3 + \sum_{i=1}^n 3i - \sum_{i=1}^n 1 = 3 \sum_{i=1}^n 3i^2 = 3T_n^2$ . Note that  $\sum_{i=1}^n 1 = n$  and using the formula for  $T_n^1$  we have  $\sum_{i=1}^n 3i = 3T_n^1 = 3 \cdot \frac{n(n+1)}{2}$ . So  $T_n^2 = \frac{1}{3} \left[ \frac{2n^3}{2} + \frac{3n(n+1)}{2} - \frac{2n}{2} \right] = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$ . If we are just given the formula we can show it is correct by induction.