The triangular numbers are the number of items in a triangular stack. We write this as T_n if there are n items in the bottom row. For example, in a stack with 8 boxes on the bottom row we have a total of $T_8 = 1 + 2 + \cdots + 8 = 36$ boxes.

$$
\begin{array}{cccccccc}
\square & & & & & & & \\
\square & \square & & & & & & \\
\square & \square & \square & & & & & \\
\square & \square & \square & \square & & & & \\
\square & \square & \square & \square & \square & & & \\
\square & \square & \square & \square & \square & \square & & \\
\square & \square & \square & \square & \square & \square & \square & \\
\square & \square & \square & \square & \square & \square & \square & \square
\end{array}
$$

Observe that the sequence $T_1, T_2, T_3, \ldots, = 1, 3, 6, 10, 15, 21, 36, \ldots$ corresponds to the third column of the binomial triangle so we might expect $T_n = \binom{n+1}{2} = \frac{(n+1)n}{2}$. This is indeed the case.

There are many elementary proofs of this fact. For example write T_n twice, once with the numbers in reverse order:

$$
\begin{array}{ccccccccccccc}
T_n & = & 1 & + & 2 & + & 3 & + & \cdots & + & n \\
T_n & = & n & + & n-1 & + & n-2 & + & \cdots & + & 1 \\
\hline
2T_n & = & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1)
\end{array}
$$

There are n terms on the right so $2T_n = n(n+1)$ or $T_n = \frac{n(n+1)}{2}$.

In order to illustrate induction we will give an proof by induction even though there are much shorter proofs.

The triangular numbers satisfy $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for $n = 1, 2, \ldots$

Proof: By induction. The formula is trivial when $n = 1$ as $T_1 = 1 = \frac{1(1+1)}{2}$. By induction we may assume that $1 + 2 + \cdots + (n-1) = \frac{(n-1)(n-1+1)}{2} = \frac{(n-1)n}{2}$. Then $T_n = (1 + 2 + \cdots + n - 1) + n = \frac{(n-1)n}{2} + n = \frac{(n-1)n}{2} + \frac{2n}{2} = \frac{n((n-1)+2}{2} = \frac{n(n+1)}{2}$. So the formula holds for n and by induction the formula holds for all $n = 1, 2, \ldots,$ $\square$

We can consider in general sums of powers of integers $T_n^k + \sum_{i=1}^n i^k$ for other powers. There are methods to work out T_n^k in terms of T_n^j for $j < k$ but there is no simple general expression for these. We will illustrate this for sums of squares. $n^3 = \sum_{i=1}^n [i^3 - (i-1)^3] = \sum_{i=1}^n [i^3 - (i^3 - 3i^2 + 3i - 1] = \sum_{i=1}^n (3i^2 - 3i + 1)$. Thus $n^3 + \sum_{i=1}^n 3i - \sum_{i=1}^n 1 = 3\sum_{i=1}^n 3i^2 = 3T_n^2$. Note that $\sum_{i=1}^n 1 = n$ and using the formula for T_n^1 we have $\sum_{i=1}^n 3i = 3T_n^1 = 3 \cdot \frac{n(n+1)}{2}$. So $T_n^2 = \frac{1}{3}\left[\frac{2n^3}{2} + \frac{3n(n+1)}{2} - \frac{2n}{2}\right] = \frac{2n^3+3n^2+n}{6} = \frac{n(n+1)(2n+1)}{6}$. If we are just given the formula we can show it is correct by induction.