Math 163 Introductory Seminar - Lehigh University - Spring 2008 - Assignment 8 Solutions
Due Monday March 31

29. Consider the variants for strong duality listed below.
B': If both problems are feasible then :
$\max\{\boldsymbol{cx}|A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$
C': If both problems are feasible then :
$\max\{\boldsymbol{cx}|A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A \geq \boldsymbol{c}\}$

(a) Prove that B' implies C'
(b) Prove that C' implies B'

Indicate clearly which is part (a) and which is part (b) in your solution.

(a) To show (B') implies (C'): Assuming the first and last LPs below are feasible we have

$$\begin{aligned}\max\{\boldsymbol{cx}|A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} &= \max\left\{\boldsymbol{cx}\middle|\left[\frac{A}{-A}\right]\boldsymbol{x} \leq \left[\frac{\boldsymbol{b}}{-\boldsymbol{b}}\right]\right\} \\ &= \min\left\{[\ \boldsymbol{u} \mid \boldsymbol{v}\ ]\left[\frac{\boldsymbol{b}}{-\boldsymbol{b}}\right] \middle| [\ \boldsymbol{u} \mid \boldsymbol{v}\ ]\left[\frac{A}{-A}\right] \geq \boldsymbol{c}, [\ \boldsymbol{u} \mid \boldsymbol{v}\ ] \geq \boldsymbol{0}\right\} \\ &= \min\{\boldsymbol{yb}|\boldsymbol{y}A \geq \boldsymbol{c}\}\end{aligned}$$

The first and third equalities follow from basic manipulations and letting $\boldsymbol{y} = \boldsymbol{u} - \boldsymbol{v}$ for the third equality. The second follows from (B').

(b) To show (C') implies (B'): Assuming the first and last LPs below are feasible we have

$$\begin{aligned}\max\{\boldsymbol{cx}|A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} &= \max\left\{[\ \boldsymbol{c} \mid \boldsymbol{0}\ ]\left[\frac{\boldsymbol{x}}{\boldsymbol{s}}\right] \middle| [\ A \mid I\ ]\left[\frac{\boldsymbol{x}}{\boldsymbol{s}}\right] = \boldsymbol{b}, \left[\frac{\boldsymbol{x}}{\boldsymbol{s}}\right] \geq \boldsymbol{0}\right\} \\ &= \min\{\boldsymbol{yb}|\boldsymbol{y}[\ A \mid I\ ] \geq [\ \boldsymbol{c} \mid \boldsymbol{0}\ ]\} \\ &= \min\{\boldsymbol{yb}|\boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}\end{aligned}$$

The first and third equalities follow from basic manipulations. The second follows from (C').

30. Consider the following version of strong duality:
A': If both problems are feasible then : $\max\{\boldsymbol{cx}|A\boldsymbol{x} \leq \boldsymbol{b}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$
Use this to prove the following version of Farkas' Lemma:
A: Exactly one of the following holds: (I) $A\boldsymbol{x} \leq \boldsymbol{b}$, has a solution $\boldsymbol{x}$
(II) $\boldsymbol{y}A = \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{yb} < 0$ has a solution $\boldsymbol{y}$

Hints: Consider $A\boldsymbol{x} \leq \boldsymbol{b}$ in the statement of Farkas' lemma. Introduce a new variable z and subtract it from each inequality and maximize z subject to these new constraints. Write down what new A', $\boldsymbol{c}', \boldsymbol{x}', \boldsymbol{b}'$ are for this. Be careful to include the coefficients for the original $\boldsymbol{x}$ in $\boldsymbol{c}'$. Explain why this new problem is feasible and why $A\boldsymbol{x} \leq \boldsymbol{b}$ has a solution if and only if the maximum is at least 0. So then, if $A\boldsymbol{x} \leq \boldsymbol{b}$ does not have a solution then the maximum is negative and by duality so is the minimum in the dual problem. Write down the dual for the new problem with the A', $\boldsymbol{c}', \boldsymbol{x}', \boldsymbol{b}'$ and show that a negative solution for this provides a solution to $\boldsymbol{y}A = \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{yb} < 0$.

Note first that it is easy to show that at most one of (I) and (II) holds as follows: If both (I) and (II) hold then using $\boldsymbol{y} \geq \boldsymbol{0}$ for the third inequality and basic substitutions for the others we have

$$0 = \boldsymbol{0x} = (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) \leq \boldsymbol{yb} < 0$$

a contradiction.

Use a new variable z. For $i = 1, 2, \ldots, m$ note that $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ has a solution if and only if $(\sum_{j=1}^{n} a_{ij}x_j) + z \leq b_i$ has a solution with the value assigned to z at least 0. Then using $\mathbf{1}$ to denote a column vector (with m rows) with every entry 1 we have that $A\boldsymbol{x} \leq \boldsymbol{b}$ is feasible if and only if $\left[\begin{array}{c|c} A & \mathbf{1} \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ \hline z \end{array}\right] \leq \boldsymbol{b}$ has a solution with $z \geq 0$. So $A\boldsymbol{x} \leq \boldsymbol{b}$ is feasible if and only if $\max\left\{\left[\begin{array}{c|c} \mathbf{0} & 1 \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ \hline z \end{array}\right] \mid \left[\begin{array}{c|c} A & \mathbf{1} \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ \hline z \end{array}\right] \leq \boldsymbol{b}\right\}$ is at least 0. Note that this linear program is feasible since taking $\boldsymbol{x} = \boldsymbol{0}$ and z to be at most the smallest of the b_i is feasible.

Now, to show that at least one of (I) or (II) holds, assume that (I) does not hold and show that in this case (II) holds. From the previous paragraph, if (I) does not hold the maximum is negative in the linear program $\max\{\boldsymbol{c'x'} | A'\boldsymbol{x'} \leq \boldsymbol{b}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A' = \boldsymbol{c'}, \boldsymbol{y} \geq \boldsymbol{0}\}$ where $A' = \left[\begin{array}{c|c} A & \mathbf{1} \end{array}\right]$, $\boldsymbol{x'} = \left[\begin{array}{c} \boldsymbol{x} \\ \hline z \end{array}\right]$ and $\boldsymbol{c'} = \left[\begin{array}{c|c} \mathbf{0} & 1 \end{array}\right]$. So for $\boldsymbol{y}^*$ attaining the minimum we have $\boldsymbol{y}^*\boldsymbol{b} < 0$ since the minimum is negative. From dual feasibility we also have $\boldsymbol{y}^* \left[\begin{array}{c|c} A & \mathbf{1} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{0} & 1 \end{array}\right]$ and $\boldsymbol{y}^* \geq \boldsymbol{0}$. This gives $\boldsymbol{y}^*A = \boldsymbol{0}, \boldsymbol{y}^* \geq \boldsymbol{0}$ with $\boldsymbol{y}^*\boldsymbol{b} < 0$ and hence (II) has a solution.

31. Given pairs of data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ consider approximating lines of the form $y = mx + b$. The error e_i for the i^{th} pair is the distance between y_i and the height (y value) of the line at x_i. This is $e_i = y_i - (mx_i + b)$. If we consider the equations $b + x_i m = y_i$ for $i = 1, 2, \ldots, n$ in the variables b and m we can think of this as a system of equations $A\boldsymbol{x} = \boldsymbol{b}$ where $A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} b \\ m \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. The best least squares approximation for this system (which gives the intercept b and slope m of the best least squares line for the data) is the solution to the normal equations $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$. Determine $A^T A$ (a 2×2 matrix) and $A^T\boldsymbol{b}$ (a 2×1 matrix). The entries will be sums of terms involving the x_i and y_i. Write these, first using $\sum$ notation and then simplify the notation using $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, $\overline{y} = \frac{\sum_{i=1}^{n} y_i}{n}$, $\overline{x}^2 = \frac{\sum_{i=1}^{n} x_i^2}{n}$ and $\overline{xy} = \frac{\sum_{i=1}^{n} x_i y_i}{n}$. Write down the system of 2 equations in the 2 unknowns m, b with coefficients in terms of the expressions in the previous sentence. Solve this system first for m and then determine b in terms of m (and the coefficients). Determine the least squares line for the points $(0, 2), (1, 1), (3, 4)$ using your results.

All sums are indexed $\sum_{i=1}^{n}$ so we will drop the indices to make notation less cluttered.

$$A^TA = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} = n \begin{bmatrix} 1 & \overline{x} \\ \overline{x} & \overline{x^2} \end{bmatrix} \text{ and}$$

$$A^T\boldsymbol{b} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix} = n \begin{bmatrix} \overline{y} \\ \overline{xy} \end{bmatrix}.$$

So with $\boldsymbol{x} = \begin{bmatrix} b \\ m \end{bmatrix}$ we get that $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ becomes (after canceling the n) $\begin{bmatrix} 1 & \overline{x} \\ \overline{x} & \overline{x^2} \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} \overline{y} \\ \overline{xy} \end{bmatrix}$. Writing this as a system of 2 equations in the two unknowns b and m this is

$\begin{array}{ccccc} b & + & \overline{x}m & = & \overline{y} \\ \overline{x}b & + & \overline{x^2}m & = & \overline{xy} \end{array}$. To solve this for m, multiply the first equation by $\overline{x}$ and then subtract the second to get $0 + \overline{x}^2 m - \overline{x^2} m = \overline{x}\,\overline{y} - \overline{xy}$. From this we get $m = \frac{\overline{x}\,\overline{y} - \overline{xy}}{\overline{x}^2 - \overline{x^2}}$. Then, writing b in terms of m we have $b = \overline{y} - \overline{x}m = \overline{y} - \overline{x}\,\frac{\overline{x}\,\overline{y} - \overline{xy}}{\overline{x}^2 - \overline{x^2}}$.

For the points $(0, 2), (1, 1), (3, 4)$ we have $\overline{x} = (0 + 1 + 3)/3 = 4/3$, $\overline{y} = (2 + 1 + 4)/3 = 7/3$, $\overline{x^2} = (0 + 1 + 9)/3 = 10/3$ and $\overline{xy} = (0 + 1 + 12)/3 = 13/3$. Hence the least squares line for these points has slope $m = \frac{\frac{4}{3}\frac{7}{3} - \frac{13}{3}}{(\frac{4}{3})^2 - \frac{10}{3}} = \frac{11}{14}$ and intercept $b = \frac{7}{3} - \frac{4}{3}\frac{11}{14} = \frac{9}{7}$.