Math 163 Introductory Seminar - Lehigh University - Spring 2008 - Assignment 6 Solutions
Due Wednesday February 27

18. Apply Fourier-Motzkin elimination to the following systems. Using this method, either determine one solution or give a certificate showing the original system is inconsistent. Use Fourier-Motzkin elimination to get your answers, showing how it could work on much larger systems.

$$
\textbf{(a)}\quad \begin{array}{rcrcr} x_1 & + & x_2 & \leq & 3 \\ 2x_1 & - & 8x_2 & \leq & 8 \\ -3x_1 & + & 9x_2 & \leq & -15 \end{array} \qquad \textbf{(b)}\quad \begin{array}{rcrcr} x_1 & + & x_2 & \leq & 3 \\ 2x_1 & - & 8x_2 & \leq & 4 \\ -3x_1 & + & 9x_2 & \leq & -15 \end{array}
$$

We will eliminate the variable x_1. Rewriting we get the following 'equivalent' systems (where equivalent means either both have solutions or both do not):

For (a)

$$
\begin{array}{lcl} x_1 + x_2 & \leq & 3 \\ 2x_1 - 8x_2 & \leq & 8 \\ -3x_1 + 9x_2 & \leq & -15 \end{array} \Leftrightarrow \begin{array}{lcl} x_1 & \leq & 3 - x_2 \\ x_1 & \leq & 4 + 4x_2 \\ 5 + 3x_2 & \leq & x_1 \end{array} \Leftrightarrow \begin{array}{lcl} 5 + 3x_2 \leq & 3 - x_2 \\ 5 + 3x_2 & \leq & 4 + 4x_2 \end{array} \Leftrightarrow \begin{array}{lcl} 4x_2 \leq & -2 \\ -x_2 & \leq & -1 \end{array}
$$

The last system is inconsistent as seen by the multipliers $(1/4, 1)$. (Multiply the first inequality by $1/4$ and the second by 1 and combine to get $0 \leq -3/2$.) This correspond to multipliers $(1/4, 1)$ in the third system, multipliers $(1/4, 1, 1 + 1/4) = (1/4, 1, 5/4)$ in the second system and $(1/4, 1/2, 5/12)$ in the original, yielding the same inconsistency $0 \leq -3/2$.

For (b)

$$
\begin{array}{lcl} x_1 + x_2 & \leq & 3 \\ 2x_1 - 8x_2 & \leq & 4 \\ -3x_1 + 9x_2 & \leq & -15 \end{array} \Leftrightarrow \begin{array}{lcl} x_1 & \leq & 3 - x_2 \\ x_1 & \leq & 2 + 4x_2 \\ 5 + 3x_2 & \leq & x_1 \end{array} \Leftrightarrow \begin{array}{lcl} 5 + 3x_2 \leq & 3 - x_2 \\ 5 + 3x_2 & \leq & 2 + 4x_2 \end{array} \Leftrightarrow \begin{array}{lcl} 4x_2 \leq & -2 \\ -x_2 & \leq & -3 \end{array}
$$

The last system is inconsistent as seen by the multipliers $(1/4, 1)$. (Multiply the first inequality by $1/4$ and the second by 1 and combine to get $0 \leq -7/2$.) This correspond to multipliers $(1/4, 1)$ in the third system, multipliers $(1/4, 1, 1 + 1/4) = (1/4, 1, 5/4)$ in the second system and $(1/4, 1/2, 5/12)$ in the original, yielding the same inconsistency $0 \leq -7/2$.

19. Prove the following version of weak duality:
If both problems are feasible then $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}\} \leq \min\{\boldsymbol{yb} | \boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$.
Give *two* proofs. One using matrix notation and one using $\sum$ notation.

For any feasible $\boldsymbol{x}^*$ and $\boldsymbol{y}*$ we have

$$
\boldsymbol{cx}^* = (\boldsymbol{y}^* A)\boldsymbol{x}^* = \boldsymbol{y}^*(A\boldsymbol{x}^*) \leq \boldsymbol{y}^*\boldsymbol{b}
$$

where the first inequality follows since $\boldsymbol{y}^* A = \boldsymbol{c}$, the second inequality follows from associativity and the inequality follows since $\boldsymbol{y}^* \geq \boldsymbol{0}$ and $A\boldsymbol{x}^* \leq \boldsymbol{b}$.

We give an alternate proof using summation notation. For any feasible x_i^* and y_j^* we have $\sum_{j=1}^{n} c_j x_j^* = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{i,j} y_i^* \right) x_j^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} x_j^* \right) y_i^* \leq \sum_{i=1}^{m} b_i y_i^*$.

20. Consider the following statement: If the dual is unbounded then the primal is infeasible. Prove this for the versions of linear programming problems in problem 19 by first stating the contrapositive and then proving that. You may use the result of problem 19.

The contrapositive is: If the primal is feasible then the dual is bounded. To show this note that since the primal is feasible there is a feasible solution $\boldsymbol{x}^*$. That is, $\boldsymbol{x}^*$ is such that $A\boldsymbol{x}^* \leq \boldsymbol{b}$. From weak duality, for any feasible $\boldsymbol{cx}^* \leq \boldsymbol{y}^*\boldsymbol{b}$. Thus the dual is bounded below by $\boldsymbol{cx}^*$.

21. Consider the linear programming problem:

$$
\begin{array}{lrcrcrcr}
\max & x_1 & + & 2x_2 & + & 3x_3 & & \\
\text{s.t.} & 4x_1 & - & 5x_2 & + & 6x_3 & = & 7 \\
 & 8x_1 & & & + & 9x_3 & = & 10 \\
 & x_1 & + & x_2 & + & x_3 & \leq & 0 \\
 & x_1 & - & x_2 & + & 13x_3 & \leq & 1 \\
 & x_1 & , & x_2 & & & \geq & 0
\end{array}
$$

(a) Write down an equivalent problem that is in the form $\max\{\boldsymbol{cx} | A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$. Write out the equations as above. Also give the matrix A and vectors $\boldsymbol{b}, \boldsymbol{c}$ for this.
(b) Write down the duals to both the original problem and the problem in part (a).

(a)

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 2x_2 & + & 3x_3^+ & - & 3x_3^- & & & & & & \\
\text{s.t.} & 4x_1 & - & 5x_2 & + & 6x_3^+ & - & 6x_3^- & & & & & = & 7 \\
 & 8x_1 & & & + & 9x_3^+ & - & 9x_3^- & & & & & = & 10 \\
 & x_1 & + & x_2 & + & x_3^+ & - & x_3^- & + & s_1 & & & = & 0 \\
 & x_1 & - & x_2 & + & 13x_3^+ & - & 13x_3^- & & & + & s_2 & = & 1 \\
 & x_1 & , & x_2 & , & x_3^+ & , & x_3^- & , & s_1 & , & s_2 & \geq & 0
\end{array}
$$

$$
A = \begin{bmatrix} 4 & -5 & 6 & -6 & 0 & 0 \\ 8 & 0 & 9 & -9 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 13 & -13 & 0 & 1 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3^+ \\ x_3^- \\ s_1 \\ s_2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 7 \\ 10 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 1 & 2 & 3 & -3 & 0 & 0 \end{bmatrix}
$$

(b)

$$
\begin{array}{lrcrcrcrcr}
\min & 7y_1 & + & 10y_2 & & & + & y_4 & & \\
\text{s.t.} & 4y_1 & + & 8y_2 & + & y_3 & + & y_4 & \geq & 1 \\
 & -5y_1 & & & + & y_3 & - & y_4 & \geq & 2 \\
 & 6y_1 & + & 9y_2 & + & y_3 & + & 13y_4 & = & 3 \\
 & & & & & y_3 & , & y_4 & \geq & 0
\end{array}
$$

$$
\begin{array}{lrcrcrcrcr}
\min & 7u_1 & + & 10u_2 & & & + & u_4 & & \\
\text{s.t.} & 4u_1 & + & 8u_2 & + & u_3 & + & u_4 & \geq & 1 \\
 & -5u_1 & & & + & u_3 & - & u_4 & \geq & 2 \\
 & 6u_1 & + & 9u_2 & + & u_3 & + & 13u_4 & \geq & 3 \\
 & -6u_1 & - & 9u_2 & - & u_3 & - & 13u_4 & \geq & -3 \\
 & & & & & u_3 & & & \geq & 0 \\
 & & & & & & & u_4 & \geq & 0
\end{array}
$$

22. Prove the equivalence of B and C:

B: Exactly one of the following holds:
(I) $A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ has a solution $\boldsymbol{x}$ *or* (II) $\boldsymbol{y}A \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0$ has a solution $\boldsymbol{y}$

C: Exactly one of the following holds:
(I) $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ has a solution $\boldsymbol{x}$ *or* (II) $\boldsymbol{y}A \geq \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0$ has a solution $\boldsymbol{y}$

Note - there are at least two ways to take care of the 'at most one of the systems has a solution' part of the statements. While it is a bit redundant we will show both ways below. First we show it directly and we also show it by the equivalent systems. If the 'at most one system holds' is shown first then only the $\Leftarrow$'s are needed for the equivalent systems.

First we note that for each it is easy to show that at most one of the systems holds.

If both IB and IIB hold then

$$0 = \boldsymbol{0}\boldsymbol{0} \leq (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) \leq \boldsymbol{y}\boldsymbol{b} < 0$$

a contradiction. We have used $\boldsymbol{y} \geq \boldsymbol{0}$ in the second $\leq$.

If both IC and IIC hold then

$$0 = \boldsymbol{0}\boldsymbol{0} \leq (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) = \boldsymbol{y}\boldsymbol{b} < 0$$

a contradiction.

So for the remainder we will seek to show *at least one of the following holds.*

$(B \Rightarrow C)$: Note the following equivalences.

$$
\text{(IC)} \quad \begin{array}{c} A\boldsymbol{x} = \boldsymbol{b} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} A\boldsymbol{x} \leq \boldsymbol{b} \\ -A\boldsymbol{x} \leq -\boldsymbol{b} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \begin{bmatrix} A \\ -A \end{bmatrix} \boldsymbol{x} \leq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \text{(IC')}
$$

and

$$
\text{(IIC)} \quad \begin{array}{c} \boldsymbol{y}A \geq \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} (\boldsymbol{u} - \boldsymbol{v})A \geq \boldsymbol{0} \\ (\boldsymbol{u} - \boldsymbol{v})\boldsymbol{b} < 0 \\ \boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \begin{bmatrix} A \\ -A \end{bmatrix} \geq \boldsymbol{0} \\ \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} < 0 \\ \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \geq \boldsymbol{0} \end{array} \quad \text{(IIC')} \ .
$$

In the second line, given $\boldsymbol{y}$ one can easily pick non-negative $\boldsymbol{u}, \boldsymbol{v}$ such that $\boldsymbol{y} = \boldsymbol{u} - \boldsymbol{v}$ so the first $\Leftrightarrow$ in the second line does hold.

Applying B, we get that exactly one of (IC') and (IIC') has a solution. The equivalences then show that exactly one of (IC) and (IIC) has a solution.

$(C \Rightarrow B)$: Note following equivalences.

$$\text{(IB)} \quad \begin{array}{c} A\boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} A\boldsymbol{x} + I\boldsymbol{s} = \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{s} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \left[\begin{array}{cc} A & I \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ \boldsymbol{s} \end{array}\right] = \boldsymbol{b} \\ \left[\begin{array}{c} \boldsymbol{x} \\ \boldsymbol{s} \end{array}\right] \geq \boldsymbol{0} \end{array} \quad \text{(IB')}$$

and

$$\text{(IIB)} \quad \begin{array}{c} \boldsymbol{y}A \geq \boldsymbol{0} \\ \boldsymbol{y} \geq \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y}A \geq \boldsymbol{0} \\ \boldsymbol{y}I \geq \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y} \left[\begin{array}{cc} A & I \end{array}\right] \geq \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \text{(IIB')} \; .$$

Applying C, we get that exactly one of (IB') and (IIB') has a solution. The equivalences then show that exactly one of (IB) and (IIB) has a solution.

23. Prove the equivalence of A' and B':

A': If both problems are feasible then : $\max\{\boldsymbol{c}\boldsymbol{x} | A\boldsymbol{x} \leq \boldsymbol{b}\} = \min\{\boldsymbol{y}\boldsymbol{b} | \boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$

B': If both problems are feasible then : $\max\{\boldsymbol{c}\boldsymbol{x} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{y}\boldsymbol{b} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$

To show (A') implies (B'): Assuming the first and last LPs below are feasible we have

$$\begin{aligned} \max\{\boldsymbol{c}\boldsymbol{x} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} &= \max\left\{\boldsymbol{c}\boldsymbol{x} | \left[\begin{array}{c} A \\ \hline -I \end{array}\right] \boldsymbol{x} \leq \left[\begin{array}{c} \boldsymbol{b} \\ \hline \boldsymbol{0} \end{array}\right]\right\} \\ &= \min\left\{\left[\begin{array}{c|c} \boldsymbol{u} & \boldsymbol{v} \end{array}\right] \left[\begin{array}{c} \boldsymbol{b} \\ \hline \boldsymbol{0} \end{array}\right] | \left[\begin{array}{c|c} \boldsymbol{u} & \boldsymbol{v} \end{array}\right] \left[\begin{array}{c} A \\ \hline -I \end{array}\right] = \boldsymbol{c}, \left[\begin{array}{c|c} \boldsymbol{u} & \boldsymbol{v} \end{array}\right] \geq \boldsymbol{0}\right\} \\ &= \min\left\{\boldsymbol{y}\boldsymbol{b} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\right\} \end{aligned}$$

The first and third equalities follow from basic manipulations. The second follows from (A').

To show (B') implies (A'): Assuming the first and last LPs below are feasible we have

$$\begin{aligned} \max\left\{\boldsymbol{c}\boldsymbol{x} | A\boldsymbol{x} \leq \boldsymbol{b}\right\} &= \max\left\{\left[\begin{array}{c|c} \boldsymbol{c} & -\boldsymbol{c} \end{array}\right] \left[\begin{array}{c} \boldsymbol{u} \\ \hline \boldsymbol{v} \end{array}\right] | \left[\begin{array}{c|c} A & -A \end{array}\right] \left[\begin{array}{c} \boldsymbol{u} \\ \hline \boldsymbol{v} \end{array}\right] \leq \boldsymbol{b}, \left[\begin{array}{c} \boldsymbol{u} \\ \hline \boldsymbol{v} \end{array}\right] \geq \boldsymbol{0}\right\} \\ &= \min\left\{\boldsymbol{y}\boldsymbol{b} | \boldsymbol{y}\left[\begin{array}{c|c} A & -A \end{array}\right] \geq \left[\begin{array}{c|c} \boldsymbol{c} & -\boldsymbol{c} \end{array}\right], \boldsymbol{y} \geq \boldsymbol{0}\right\} \\ &= \min\{\boldsymbol{y}\boldsymbol{b} | \boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\} \end{aligned}$$

The first and third equalities follow from basic manipulations. The second follows from (B').