Math 163 Introductory Seminar - Lehigh University - Spring 2008 - Assignment 3 Solutions
Due Friday February 8

9. Consider the system of equations $\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$ in the variables x_1 and x_2.

Give a solution describing x_1 and x_2 in terms of the given values $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ and state conditions on these values for which the (unique) solution exists. (Show work done in obtaining the solution. Do not just state it.)

Multiply the first equation by $-a_{21}$ and the second by a_{11} to get
$$\begin{array}{rcrcl} -a_{21}a_{11}x_1 & - & a_{21}a_{12}x_2 & = & -a_{21}b_1 \\ a_{11}a_{21}x_1 & + & a_{11}a_{22}x_2 & = & a_{11}b_2 \end{array}.$$
Adding we get $0x_1 + (a_{11}a_{22} - a_{12}a_{21})x_2 = (a_{11}b_2 - a_{21}b_1)$ and thus $x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$.

Similarly multiply the first equation by a_{22} and the second by $-a_{12}$ to get
$$\begin{array}{rcrcl} a_{22}a_{11}x_1 & + & a_{22}a_{12}x_2 & = & a_{22}b_1 \\ -a_{12}a_{21}x_1 & - & a_{12}a_{22}x_2 & = & -a_{12}b_2 \end{array}.$$
Adding we get $(a_{11}a_{22} - a_{12}a_{21})x_1 + 0x_2 = (a_{22}b_1 - a_{12}b_2)$ and thus $x_1 = \frac{a_2b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$.

Both have the same denominator. When this is not 0 we get a unique solution. Thus we get a unique solution when $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

10. Consider the answer to problem 9. When the conditions fail, give conditions on $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ so that
(i) the system has infinitely many solutions
(ii) the system has no solution.

Rewriting the equations we get $x_2 = \frac{-a_{11}}{a_{12}}x_1 + \frac{b_1}{a_{12}}$ and $x_2 = \frac{-a_{21}}{a_{22}}x_1 + \frac{b_2}{a_{22}}$. When $a_{11}a_{22} - a_{12}a_{21} = 0$ we have $\frac{-a_{11}}{a_{12}} = \frac{-a_{21}}{a_{22}}$ and thus the lines have the same slope. If the lines are parallel there are no solutions and if both equations give the same line there are infinitely many solutions. They are the same when the intercepts are the same, $\frac{b_1}{a_{12}} = \frac{b_2}{a_{22}}$ and they are different parallel lines when the intercepts differ, $\frac{b_1}{a_{12}} \neq \frac{b_2}{a_{22}}$. So we get infinitely many solutions when $b_1a_{22} = b_2a_{12}$ and no solutions when $b_1a_{22} \neq b_2a_{12}$. Note that had we written the lines in terms of x_1 instead of x_2 we would have stated that there are infinitely many solutions when $b_1a_{21} = b_2a_{11}$ and no solutions when $b_1a_{21} \neq b_2a_{11}$. With $a_{11}a_{22} - a_{12}a_{21} = 0$ these two versions are easily seen to be equivalent.