Math 163 Introductory Seminar - Lehigh University - Spring 2008 - Assignment 2 Solutions
Due Monday January 28

4. Let G be a bipartite graph with bipartition X, Y. Hall's condition is $|T| \leq |N(T)|$ for all $T \subseteq X$. Prove that when $|X| = |Y|$ this is a necessary condition for G to have a perfect matching.

We need to show that if G has a perfect matching then $|T| \leq |N(T)|$ for all $T \subseteq X$. Label the vertices so that the matching is $x_1y_1, x_2y_2, \ldots, x_ny_n$. Then, for any $T = \{x_i | i \in I\}$ we have that $\{y_i | i \in I\} \subseteq N(T)$ because the edges $x_1y_1, x_2y_2, \ldots, x_ny_n$ are all in G. Thus $|T| = |\{x_i | i \in I\}| = |\{y_i | i \in I\}| \leq |N(T)|$.

5. Let G be a bipartite graph with bipartition X, Y having $|X| = |Y|$. Prove that the sufficiency of Hall's condition for a perfect matching implies the sufficiency of the marriage condition. That is prove that the statement: (If G does not have a perfect matching then there is a $T \subseteq X$ such that $|T| > |N(T)|$) implies the statement (If G does not have a perfect matching then G has a vertex cover C with $|C| < n$).

If G does not have a perfect matching then by Hall's Theorem there there is $T \subseteq X$ with $|T| > |N(T)|$. By the definition of neighborhood there are no edges between T and $Y - N(T)$. Thus every edge has an end in $X - T$ or $Y - (Y - N(T)) = N(T)$ (or both). So $C = (X - T) \cup N(T)$ is a vertex cover. Using $|N(T)| < |T|$ and $|X - T| + |T| = |X| = n$ we get $|C| = |X - T| + |N(T)| < (n - |T|) + |T| = n$ So $C = (X - T) \cup N(T)$ is a vertex cover with size less than n.

For completeness here also is a solution to problem 1 using this notation. That is, a proof that the sufficiency of the marriage condition implies the sufficiency of Hall's condition:

If G does not have a perfect matching then by the marriage theorem there is a vertex cover $C = R \cup S$ with $R \subseteq X$ and $S \subseteq Y$ with $|C| < n$. Since $R \cup S$ is a vertex cover there are no edges between $X - R$ and $Y - S$. Thus the neighborhood of $X - R$ is contained in $Y - (Y - S) = S$. That is $N(X - R) \subseteq S$. Then $|N(X - R)| \leq |S|$. Using also $n > |C| = |R| + |S|$ and we get $|N(X - R)| \leq |S| < n - |R| = |X - R|$. Thus $T = X - R$ has $|T| > |N(T)|$.

6. Use induction to prove that the Fibonacci numbers satisfy $\sum_{i=0}^{n-1} F_{2i+1} = F_{2n}$.

We show this for $n = 1, 2, \ldots,$ by induction. For $n = 1$ the formula reduces to $F_1 = F_2$ which holds since both are 1. For $n \geq 2$ we have $\sum_{i=0}^{n-1} F_{2i+1} = \left(\sum_{i=1}^{n-2} F_{2i+1}\right) + F_{2(n-1)+1} = F_{2(n-1)} + F_{2(n-1)+1} = F_{2n-2} + F_{2n-1} = F_{2n}$. The second equality follows by induction and the last by the Fibonacci recurrence (and the others are elementary algebra). Thus by induction the formula holds for $n = 1, 2 \ldots$.

7. Prove that the Arithmetic-Geometric mean inequality implies the Geometric-Harmonic mean inequality. That is, for positive numbers $y_1, y_2, \ldots, y_m$ the inequality $\frac{y_1+y_2+\cdots+y_n}{n} \geq (y_1y_2\cdots y_n)^{1/n}$ implies $(y_1y_2\cdots y_n)^{1/n} \geq \frac{n}{\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}}$.
Hint - think about reciprocals.

For all of the following we assume that the x_i and y_i are positive numbers. Use the Arithmetic-Geometric mean inequality $\frac{x_1+x_2+\cdots+x_n}{n} \geq (x_1x_2\cdots x_n)^{1/n}$ to show the Geometric-

Harmonic mean inequality $(y_1 y_2 \cdots y_n)^{1/n} \geq \frac{n}{\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}}$. Let $x_i = \frac{1}{y_i}$ for $i = 1, 2, \ldots, n$. Then applying the Arithmetic-Geometric mean inequality substituting $x_i = \frac{1}{y_i}$ we get $\frac{\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}}{n} \geq \left(\frac{1}{y_1}\frac{1}{y_2}\cdots\frac{1}{y_n}\right)^{1/n}$. Cross multiplying yields $(y_1 y_2 \cdots y_n)^{1/n} \geq \frac{n}{\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}}$.

8. Derive a formula for $\sum_{i=1}^n i^3$ as follows. Note that $n^4 = \sum_{i=1}^n (i^4 - (i-1)^4)$ and then expand the $(i-1)^4$ term. The expression being summed now has terms involving i, i^2, i^3. Use known formulas for $\sum_{i=1}^n i^2$, $\sum_{i=1}^n i$, $\sum_{i=1}^n 1$ to get a formula for $\sum_{i=1}^n i^3$.

Use the properties of a telescoping series for the first equality and the expansion of $(i-1)^4$ to get $n^4 = \sum_{i=1}^n (i^4 - (i-1)^4) = \sum_{i=1}^n (i^4 - (i^4 - 4i^3 + 6i^2 - 4i + 1)) = 4\sum_{i=1}^n i^3 - 6\sum_{i=1}^n i^2 + 4\sum_{i=1}^n i - \sum_{i=1}^n 1$. Then moving terms and substituting $\sum_{i=1}^n 1 = n$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ we get $4\sum_{i=1}^n i^3 = n^4 + 6\frac{(n)(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n = n^4 + n(n+1)(2n+1) - 2n(n+1) + n = n(n^3 + (2n^2 + 3n + 1) - (2n+2) + 1) = n(n^3 + 2n^2 + n) = n^2(n^2 + 2n + 1) = n^2(n+1)^2$. Thus $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$.