Math 242 fall 2008 notes on problem session for week of 9-8-08

This is a short overview of problems that we covered.

1. Prove that $A^T A$ is symmetric. Using $(BC)^T = C^T B^T$ and $(A^T)^T = A$ we get $(A^T A)^T = A^T (A^T)^T = A^T A$. Thus $A^T A$ is symmetric (since it is equal to its transpose).

2. Prove $(AB)^T = B^T A^T$. Assume that A is $m \times n$ and B is $n \times p$.

   Use subscripts like A_{ij} to indicate entries in matrix A. We need to show that $(AB)^T_{ij} = (B^T A^T)_{ij}$. From the definition of transpose, the ij entry of the transpose is the ji entry of the original. Use this and the definition of matrix multiplication.

   $(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} A^T_{kj} B^T_{ik} = \sum_{k=1}^{n} B^T_{ik} A^T_{kj} = (B^T A^T)_{ij}$.

3. The trace of a square matrix is the sum of its diagonal elements. Writing $Tr(A)$ for the trace of an $n \times n$ matrix this is $Tr(A) = \sum_{i=1}^{n} A_{ii}$. Prove that $Tr(AB) = Tr(BA)$ when both are defined. Assume that A is $m \times n$ then B is $n \times m$ in order for both to be defined. Using the definition of trace and matrix multiplication we get

   $Tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{m} A_{ik} B_{ki} = \sum_{k=1}^{m} \sum_{i=1}^{n} B_{ki} A_{ik} = \sum_{k=1}^{m} (BA)_{kk} = Tr(BA)$.

4. Find an equation for the plane $ax+by+cz = d$ through the points $(0, 2, -1), (-2, 4, 3), (2, -1, -3)$. Substituting for x, y, z from each of the three points we get three equations

$$\begin{array}{rcrcrcrcl} & & 2b & - & c & - & d & = & 0 \\ -2a & + & 4b & + & 3c & - & d & = & 0 \\ 2a & - & b & - & 3c & - & d & = & 0 \end{array}$$

   We get solutions $a = 4d/3, b = 2d/3, c = d/3, d = d$ for any choice of d. Setting $d = 3$ yields the plane $4x + 2y + z = 3$.

5. Let $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some scalar λ. Prove that $A^k\boldsymbol{x} = \lambda^k\boldsymbol{x}$. Use induction. The case $k = 1$ is given. For $k > 1$, using the fact that scalars commute with matrices and the induction hypothesis $A^{k-1}\boldsymbol{x} = \lambda^{k-1}\boldsymbol{x}$ we get $A^k\boldsymbol{x} = AA^{k-1}\boldsymbol{x} = A(\lambda^{k-1}\boldsymbol{x}) = \lambda^{k-1}A\boldsymbol{x} = \lambda^{k-1}(\lambda\boldsymbol{x}) = \lambda^k\boldsymbol{x}$. Hence by induction the result holds for all positive integers k.