Math 242 fall 2008 notes on problem session for week of 9-30-08
This is a short overview of problems that we covered.

1. For each of the following sets ask the following: Does it span $\mathbb{R}^3$? Is it linearly independent? Is it a basis.

$S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} \right\}$ Does S_1 span $\mathbb{R}^3$? Is S_1 a basis for

$$S_2 = \left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 6 \end{pmatrix} \right\}$$

$$S_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \right\}$$

We solved these and discussed the general approach to such problems.
The general approach for such questions is as follows:

For $T = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$ form the $\times n$ matrix A which has i^{th} equal to v_i and determine U for a factorization $PA = LU$. We describe how to answer these and then explain why the tests work.

Does T span $\mathbb{R}^m$? If U has a row of all zeroes the answer is no and if U does not the answer is yes. In particular, if $n < m$ there will be a row of zeroes and T will not span: we need at least m vectors to span $\mathbb{R}^m$.

Is T linearly independent? If U has a free variable the answer is no and if U does not the answer is yes. In particular, if $n > m$ there will be a free variable and T is not linearly independent.

Is T a basis? T must span $\mathbb{R}^m$ and be linearly independent. If $n < m$ then T does not span $\mathbb{R}^m$ and if $n > m$ then T is not linearly independent. So a necessary condition is that $n = m$. Note that in this case U has a row of zeroes if and only if it has a free variable. So we observe that $m = n$ (the number of vectors equals the dimension) spanning implies linear independence and vice-versa.

Recall that the rank of a matrix is the number of pivots. That is, the number of nonzero rows in U for a $PA = LU$ factorization. So using this terminology, T spans $\mathbb{R}^m$ if $rank(A) = m$ and it does not span if $rank(A) < m$ (and $rank(A)$ cannot be greater than the number of rows m). T is linearly independent if $rank(A) = n$ and it is linearly dependent if $rank(A) < n$ (and $rank(A)$ cannot be greater than the number of columns n).

Explanation for spanning test. We need to test if every $\boldsymbol{b} \in \mathbb{R}^m$ can be written as a linear combination of vectors in T. That is, is there a solution $\boldsymbol{x}^T = (x_1, x_2, \ldots, x_n)$ to $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 + \cdots + \boldsymbol{v}_n x_n = \boldsymbol{b}$? In matrix form this is $A\boldsymbol{x} = \boldsymbol{b}$. Consider a factorization $PA = LU$. If U has a row of all 0's then there will be some choice of $\boldsymbol{b}$ for which there is no solution. Recall that we use the factorization to solve the system by first solving $L\boldsymbol{c} = P\boldsymbol{b}$ and then attempting to solve $U\boldsymbol{x} = \boldsymbol{c}$. If U has a row of zeroes we can pick $\boldsymbol{c}$ with a nonzero corresponding to such a row. Then if we set $\boldsymbol{b} = P^{-1}L\boldsymbol{c}$ we have no solution to $A\boldsymbol{x} = \boldsymbol{b}$. If U has no row of zeroes we can always solve $U\boldsymbol{x} = \boldsymbol{c}$.

Explanation for linear independence test. We need to test if the only solution to $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 + \cdots + \boldsymbol{v}_n x_n = \boldsymbol{0}$ is trivial. In matrix form this is $A\boldsymbol{x} = \boldsymbol{0}$. Consider a factorization $PA = LU$. recall that the set of solutions to $A\boldsymbol{x} = \boldsymbol{0}$ is the same as the solutions to $U\boldsymbol{x} = \boldsymbol{0}$. There are nontrivial solutions exactly when there are free variables.

For the sets above we get the following A and factorization $A = LU$. In each case $P = I$. We only need U to answer our questions but it is useful to see the factorization.

For S_1: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So S_1 does not span $\mathbb{R}^3$ and is not linearly independent.

For S_2: $\begin{bmatrix} 1 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. So S_2 does not span $\mathbb{R}^3$ and is linearly independent.

For S_3: $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. So S_3 spans $\mathbb{R}^3$ and is linearly independent. It is a basis.

2. For each of the matrices above determine a basis for the four fundamental subspaces. (We did not actually do this for these matrices in the problem session.) Also what are the dimensions and note relations between them.

Bases for the four fundamental subspaces. Given $A = LU$ we determine bases for the four fundamental subspaces as follows. Explanations as to why these methods work will be given in another set of notes and (except for the method we use for the cokernel) on pages 116-119 of the text.

The matrix A for S_1. Here $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.

Basis for $Ker(A)$: Using back substitution in U the set of solutions to $U\boldsymbol{x} = \boldsymbol{0}$ is

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} x_4$ where x_3, x_4 can be any real numbers. Thus a basis is $\{(1,-2,1,0)^T, (2,-3,0,1)^T\}$. The dimension is 2.
Basis for $Corng(A)$: Use nonzero rows of U: a basis is $\{(1,1,1,1)^T, (0,1,2,3)^T\}$. The dimension is 2.
Basis for $Coker(A)$: Use the last row of L^{-1}: a basis is $\{(-1,-1,1)^T\}$. The dimension is 1.
Basis for $Rng(A)$: use columns of A corresponding to pivots columns of U: as basis is $\{(1,2,3)^T, (1,3,4)^T\}$. The dimension is 2.

The matrix A for S_2. Here $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -17 & 4 & 1 \end{bmatrix}$.
Basis for $Ker(A)$: There are no free variables so the kernel is trivial, $ker(A) = \{\mathbf{0}\}$ which by convention has an empty basis. The dimension is 0.
Basis for $Corng(A)$: Use nonzero rows of U: a basis is $\{(1,2)^T, (0,1)^T\}$. The dimension is 2.
Basis for $Coker(A)$: Use the last row of L^{-1}: a basis is $\{(-17,4,1)^T\}$. The dimension is 1.
Basis for $Rng(A)$: use columns of A corresponding to pivots columns of U: as basis is $\{(1,3,5)^T, (2,7,6)^T\}$. The dimension is 2.

The matrix A for S_3. Here $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (although we do not need to use it since there are no zero rows in U).
Basis for $Ker(A)$: There are no free variables so the kernel is trivial, $ker(A) = \{\mathbf{0}\}$ which by convention has an empty basis. The dimension is 0.
Basis for $Corng(A)$: Use nonzero rows of U: a basis is $\{(1,2,3)^T, (0,1,3)^T, (0,0,1)^T\}$. The dimension is 3.
Basis for $Coker(A)$: There are no zero roes in U so the cokernel is trivial, $coker(A) = \{\mathbf{0}\}$ which by convention has an empty basis. The dimension is 0.
Basis for $Rng(A)$: use columns of A corresponding to pivots columns of U: as basis is $\{(1,0,0)^T, (2,1,0)^T, (4,3,1)^T\}$. The dimension is 3.

3. We observed Theorem 2.49 in the examples of the dimensions of the subspaces. A restatement and informal explanation is given here. More formal versions and proofs can be found in the text on pages 116-119.

For and $m \times n$ matrix A and a $PA = LU$ factorization we have:

- dimension of the kernel of A is equal to the number of free variables in U (which is n minus the rank of A)
- dimension of the corange of A is equal to the number of nonzero rows in U, which is the number of pivot columns which is the rank of A.
- Thus we have the kernel and corange (i.e., the nullspace and row space) as subspaces of $\mathbb{R}^n$ and the sum of their dimensions is n.

We also have

- dimension of the cokernel of A is equal to the number of zero rows in U (which is m minus the rank of A)
- dimension of the range of A is equal to the number of pivot columns in U, which is the rank of A.
- Thus we have the cokernel and range (i.e., the left nullspace and column space) as subspaces of $\mathbb{R}^m$ and the sum of their dimensions is m.

In addition, since the corange is the range of A^T the dimension of the corange is the rank of A^T. Since we noted above that the dimension of the corange is the rank of A we have established that the rank of A and the rank of A^T are the same. For example if A is a 42×99 matrix and the rank is 30 then elementary row operations produce a U with 30 nonzero rows and 12 zero rows. Elementary row operations on the 99×42 matrix A^T will also produce 30 nonzero rows and for the transpose we will have 69 zero rows.

4. We proved Lemma 2.34: The elements $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis for vector space V if and only if every $\boldsymbol{x} \in V$ can be written *uniquely* as a linear combination of the basis elements. The 'only if' direction of the proof is in the text on page 103. The more obvious 'if' direction is not in the tex so we do it here.

   Proof of 'if': Since every $\boldsymbol{x}$ can be written uniquely in terms of the vectors, it can be written in terms of these vectors. So $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ span V. Since $\mathbf{0} = 0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_n$ and this trivial solution is unique, it is the only solution. Hence $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent. The vectors span V and are linearly independent so they are a basis.

5. We showed that elementary row operations do not change the corange (row space). An alternate proof is in the text on page 119. Informally we recognize that permuting rows and multiplying a row should not change which vectors we can get as combinations of

the rows. In addition, by adding a multiple of one row to another we are replacing a row by a combination of the rows so this should not change the row space. Formally we encode the operation on A in matrix notation as $B = EA$. In fact we will show that left multiplication by an invertible matrix does not change the corange. We complete the proof by noting that the matrices E encoding elementary row operations are invertible.

$\boldsymbol{c} \in corng(EA) \Rightarrow \boldsymbol{y}^T EA = \boldsymbol{c}^T$ for some $\boldsymbol{y} \Rightarrow \boldsymbol{z}^T A = (\boldsymbol{y}^T)A = E\boldsymbol{c}^T$ where $\mathrm{z}^T = \boldsymbol{y}^T A$. So $corng(EA) \subseteq corng(A)$.

$\boldsymbol{c} \in corng(A) \Rightarrow \boldsymbol{y}^T A = \boldsymbol{c}^T$ for some $\boldsymbol{y} \Rightarrow \boldsymbol{z}^T EA = \boldsymbol{y}^T E^{-1} EA = \boldsymbol{y}^T A = \boldsymbol{c}^T$ where $\mathrm{z}^T = \boldsymbol{y}^T E^{-1}$. So $corng(A) \subseteq corng(EA)$.

$corng(EA) \subseteq corng(A)$ and $corng(A) \subseteq corng(EA)$ implies $corng(A) = corng(EA)$.

This establishes that when $PA = LU$ we have $corng(A) = corng(U)$. Observe that the same is not true for range. In some of the examples above each column of U has 3rd coordinate 0 and hence the 3rd coordinate of every vector in $rng(U)$ is 0 while the same is not true for vectors in $rng(A)$. In particular, columns of A do not have this property.