Math 242 fall 2008 notes on problem session for week of 9-1-08
This is a short overview of problems that we covered.

1. For the matrix equation $LU = A$ as below, we started with $L$ and $U$ given and computed $A$.

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 5 & 1 \end{pmatrix} \begin{pmatrix} 6 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 1 & -1 \\ 12 & 5 & -1 \\ -18 & -12 & 11 \end{pmatrix} = A.$$

Note that we did not start with $A$ and determine $L$ and $U$. You should be able to do this. The negative of the entries in $L$ encode the elementary row operations in reducing $A$ to $U$. We add $-2$ times R1 to R2 and 3 time R1 to R3. Then with the new rows we add $-5$ times R2 to R3.

Given $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ we solved $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ by substituting $LU = A$ and $U\boldsymbol{x} = \boldsymbol{c}$ to get $L\boldsymbol{c} = L(U\boldsymbol{x}) = A\boldsymbol{x} = \boldsymbol{b}$. We first solve $L\boldsymbol{c} = \boldsymbol{b}$ for $\boldsymbol{c}$ using forward substitution then solve $U\boldsymbol{x} = c$ for $\boldsymbol{x}$ using back substitution. Doing this we solved $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 5 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ using forward substitution to get $\boldsymbol{c} = \begin{pmatrix} 2 \\ -3 \\ 24 \end{pmatrix}$. Then we solved

$\begin{pmatrix} 6 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 24 \end{pmatrix}$ using back substitution to get $\boldsymbol{x} = \begin{pmatrix} 19/6 \\ -5 \\ 12 \end{pmatrix}$.

We also noted that to find the second column of $A^{-1}$ we would solve as above except that we would use $\boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Finally we discussed the number of operations for computing $LU$ (approximately $n^2$) and the number of operations for solving $A\boldsymbol{x} = \boldsymbol{b}$ using forward and back substitution on $LU$ (approximately $n^3$). A detailed discussion of this is in the text on pages 50 and 51.

2. Given $AA^{-1} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix} \begin{pmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$. If $B$ is obtained from $A$ by multiplying the $3^{rd}$ row by 7, what is $B^{-1}$.

To answer this consider the more general setting $RS = T$. If $\hat{R}$ is obtained from $R$ by multiplying the $i^{th}$ row of $R$ by a scalar $c$ then if $\hat{R}S = \hat{T}$ we can see that $\hat{T}$ is obtained from $T$ by multiplying the $i^{th}$ row of $T$ by $c$. This follows directly from the view of matrix multiplication that the $i^{th}$ row of $T$ is the $i^{th}$ row of $R$ times $S$.

Similarly, using the view of matrix multiplication that the $j^{th}$ column of $T$ is $R$ times the $j^{th}$ column of $S$ we see that if $\tilde{S}$ is obtained from $S$ by multiplying the $j^{th}$ column of $S$ by a scalar $d$ then for $R\tilde{S} = \tilde{T}$ we obtain $\tilde{T}$ by multiplying the $j^{th}$ column of $T$ by $d$.

If $B = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 35 & 14 & -21 \end{pmatrix}$ (that is, multiply the $3^{rd}$ row of $A$ by 7 to get $B$), what is $B^{-1}$?

Using the information above we see that $BA^{-1}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$, $I_3$ with the $3^{rd}$ row multiplied by 7. So if we let $B^{-1}$ be obtained from $A^{-1}$ by multiplying the $3^{rd}$ column by $1/7$ then $BB^{-1}$ is obtained from $BA^{-1}$ by multiplying the $3^{rd}$ column by $1/7$ as we get the identity, as needed. Thus $B^{-1} = \begin{pmatrix} 8 & -1 & -3/7 \\ -5 & 1 & 2/7 \\ 10 & -1 & -4/7 \end{pmatrix}$

3. Determine $A^{-1}$ if $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 3 & 5 & 7 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Use the view of matrix multiplication that the $i^{th}$ row of $RS = T$ is the $i^{th}$ row of $R$ times $S$. That is, the $i^{th}$ row of $T$ is a weighted sum of the rows of $S$ with the weights given by the $i^{th}$ row of $R$. In particular if this row corresponds to the $r^{th}$ row of an identity matrix then the $i^{th}$ row of $T$ is the $r^{th}$ row of $S$. So For $AA^{-1} = I$ since the first row of $A$ is the first row of an identity then the first row of the product, which is I in this case equals the first row of $A^{-1}$. Similar reasoning tells us that every row of $A^{-1}$ except the $3^{rd}$ corresponds to the identity matrix. Thus we get $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 3 & 5 & 7 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ a & b & c & d & e & f \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ where $a, b, c, d, e, f$ are yet to be determined. From matrix multiplication we get $4 + 5a = 0, 3 + 5b = 0, 5c = 1, 7 + 5d = 0, 6 + 5e = 0, 9 + 5f = 0$. Solving and filling these values into the matrix we have $A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -4/5 & -3/5 & 1/5 & -7/5 & -6/5 & -9/5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

4. Determine if each of the following is True or False assuming the matrices are correct sizes for the operations to be defined and 0 indicates a zero matrix of appropriate size.

   (a) $A^2 = 0 \Rightarrow A = 0$.

   (b) $AB = 0 \Rightarrow A = 0$ or $B = 0$.

   (c) $AB = B \Rightarrow A = I$.

   (d) $AB = CA$ and $A^{-1}$ exists $\Rightarrow B = C$.

   Each of these is false. We give specific $2\times 2$ counterexamples. This implies that the statements are false for $2\times 2$ matrices. We then describe more general counterexamples to cover all possible sizes.

   For (4a), $\left(\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}\right)$ is one example. Observe here that $A$ must be square. In general, let $\boldsymbol{z}^T$ be any row vector with row sum 0. Let $A$ be a square matrix with every row $\boldsymbol{z}^T$. Then each row of $A^2$ is a weighted sum of multiples of $\boldsymbol{z}$ and since the sum of the weights is 0 the row is the row vector and $A^2 = 0$.

   For (4b), note that any counterexample to part (4a) is a counter example to (4b). Here is a counterexample with $A \neq B$: $A = \left(\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}\right)$ and $B = \left(\begin{array}{cc} 2 & 3 \\ 2 & 3 \end{array}\right)$. In general, take $A$ as in part (4a) and take $B$ to be any matrix with identical rows. Note here that $A$ and $B$ do not need to be square.

   For (4c), $A = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$ and $B = \left(\begin{array}{cc} 2 & 3 \\ 2 & 3 \end{array}\right)$ is one counter example. In general, take $A$ to be any matrix for which each row of $A$ is a row of an identity matrix (possible some rows can be the same) and take $B$ with identical rows.

   For (4c), one way to discover a counterexample is to left multiple by $A^{-1}$ to obtain $B = A^{-1}CA$. For example, $A = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) = A^{-1}$, $B = \left(\begin{array}{cc} 2 & 3 \\ 2 & 3 \end{array}\right)$ and $C = \left(\begin{array}{cc} 3 & 3 \\ 2 & 2 \end{array}\right)$. In general, take $A$ to be a permutation matrix, so $A^{-1} = A^T$ and $C$ to be any square matrix with identical rows and distinct entries. Then for $B = A^TCA$ note that $A^TC = C$ since left multiplication by a permutation matrix permutes the rows and rows of $C$ are identical. So $B = CA$ which will not be $C$ since right multiplication by a permutation matrix permutes the columns.

5. The inverse of the $2 \times 2$ matrix $A = \left(\begin{array}{cc} a & 0 \\ b & c \end{array}\right)$ is $A^{-1} = \left(\begin{array}{cc} 1/a & 0 \\ -b/ac & 1/c \end{array}\right)$ assuming $a \neq 0$ and $b \neq 0$ so that the inverse will exist. If $A, B, C$ are $n \times n$ matrices such that $A^{-1}$ and $C^{-1}$ exist, determine the inverse of the block matrix $M = \left(\begin{array}{cc} A & 0 \\ B & C \end{array}\right)$. One way to guess this is to note the pattern for the case with numerical entries and guess at a form $M^{-1} = \left(\begin{array}{cc} A_1 & 0 \\ X & C^{-1} \end{array}\right)$. Then $MM^{-1} = \left(\begin{array}{cc} A & 0 \\ B & C \end{array}\right)\left(\begin{array}{cc} A^{-1} & 0 \\ X & C^{-1} \end{array}\right) = \left(\begin{array}{cc} I_n & 0 \\ 0 & I_n \end{array}\right)$. If $BA^{-1} + CX = 0$ then this will be correct. Thus $CX = -BA^{-1}$. Left multiplying by $C^{-1}$ we get $X = -C^{-1}BA^{-1}$. So $M^{-1} = \left(\begin{array}{cc} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{array}\right)$.