Math 242 fall 2008 notes on problem session for week of 10-7-08
This is a short overview of problems that we covered.

1. Recall that a left inverse of an $m \times n$ matrix A is an $n \times m$ matrix B such that $BA = I_n$ and a right inverse is an $n \times m$ matrix C such that $AC = I_m$. Show that if A^TA is nonsingular then A has a left inverse and if AA^T is nonsingular then A has a right inverse.

   If $(A^TA)^{-1}$ exists, let $B = (A^TA)^{-1}A^T$. Then $BA = ((A^TA)^{-1}A^T)A = (A^TA)^{-1}(A^TA) = I_n$. So $(A^TA)^{-1}A^T$ is a left inverse.

   If $(AA^T)^{-1}$ exists, let $C = A^T(AA^T)^{-1}$. Then $AC = A(A^T(AA^T)^{-1}) = (AA^T)(AA^T)^{-1} = I_m$. So $A^T(AA^T)^{-1}$ is a right inverse.

2. Prove that for matrices A, B, if BA is defined then $ker(A) \subseteq ker(BA)$. (This is exercise 2.5.38.)

   $\boldsymbol{x} \in ker(A) \Rightarrow A\boldsymbol{x} = \mathbf{0} \Rightarrow (BA)\boldsymbol{x} = B(A\boldsymbol{x}) = B\mathbf{0} = \mathbf{0} \Rightarrow \boldsymbol{x} \in ker(BA)$.

3. Show that if $S = \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ span a nontrivial vector space V then there is a basis T for V contained in S. (This is exercise 2.4.22.)

   Let $dim(V) = m$. If $n = m$ then S is a basis since any spanning set of m vectors in an m dimensional vectors space is a basis. We will show that if $m > n$ then $S - \boldsymbol{v}_i$ spans V for some i. We repeat such deletions until we obtain a spanning set of size m contained in S which is a basis.

   Since $m > n$ the vectors in S are linearly dependent so we have $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \mathbf{0}$ for some $c_1, c_2, \ldots, c_n$ not all 0. By relabeling we may assume that $c_n \neq 0$. Then $\boldsymbol{v}_n = \frac{c_1}{c_n}\boldsymbol{v}_1 + \frac{c_2}{c_n}\boldsymbol{v}_2 + \cdots + \frac{c_{n-1}}{c_n}\boldsymbol{v}_{n-1}$. Given $\boldsymbol{v} \in V$ we have $\boldsymbol{v} = d_1\boldsymbol{v}_1 + d_2\boldsymbol{v}_2 + \cdots + d_n\boldsymbol{v}_n$ since S spans V. Substituting the expression for $\boldsymbol{v}_n$ we get $\boldsymbol{v} = d_1\boldsymbol{v}_1 + d_2\boldsymbol{v}_2 + \cdots + d_{n-1}\boldsymbol{v}_{n-1} + d_n(\frac{c_1}{c_n}\boldsymbol{v}_1 + \frac{c_2}{c_n}\boldsymbol{v}_2 + \cdots + \frac{c_{n-1}}{c_n}\boldsymbol{v}_{n-1}) = (d_1 + \frac{d_nc_1}{c_n})\boldsymbol{v}_1 + (d_2 + \frac{d_nc_2}{c_n})\boldsymbol{v}_2 + \cdots + (d_{n-1} + \frac{d_nc_{n-1}}{c_n})\boldsymbol{v}_{n-1}$. So $\boldsymbol{v} \in span(S - \boldsymbol{v}_n)$.

4. Show that if $T = \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are a linearly independent set of vectors in a vector space V then there is a basis for V containing T. (This is similar to exercise 2.4.24.)

   Let $dim(V) = m$. If $n = m$ then T is a basis since any independent set of m vectors in an m dimensional vectors space is a basis. We will show that if $n < m$ then adding any vector not in the span of T to T produces a new independent set. We repeat such additions until we obtain an independent set of size m containing T which is a basis.

   Pick any vector in $V - span(T)$ and call it $\boldsymbol{v}_{n+1}$. Consider solutions to $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n + c_{n+1}\boldsymbol{v}_{n+1} = \mathbf{0}$. If $c_{n+1} \neq 0$ then $\boldsymbol{v}_{n+1} = \frac{-c_1}{c_{n+1}}\boldsymbol{v}_1 + \frac{-c_2}{c_{n+1}}\boldsymbol{v}_2 + \cdots + \frac{-c_n}{c_{n+1}}\boldsymbol{v}_n$. This contradict the choice $\boldsymbol{v}_{n+1} \notin span(T)$. So $c_{n+1} = 0$ and we have $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 +$

$\cdots + c_n \boldsymbol{v}_n = \mathbf{0}$. Now since T is linearly independent $c_1 = c_2 = \cdots = c_n = 0$. Thus the only solution is the trivial solution and $T \cup \{\boldsymbol{v}_{n+1}\}$ is linearly independent.

5. 2.3.17 - Prove or give a counterexample: If $\boldsymbol{z}$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ then $\boldsymbol{w}$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{z}$. This is false. For example $(1,1,0) = 1 \cdot (1,0,0) + 1 \cdot (0,1,0) + 0 \cdot (0,0,1)$ but clearly $(0,0,1)$ is not a linear combination of $(1,0,0), (0,1,0), (1,1,0)$.

6. 2.3.29 - Prove or give a counterexample to the following: If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are elements of a vector space V and do not span V, then they are linearly independent. False. For example if any two of the $\boldsymbol{v}_i$ are identical. Another example, $(1,1,0), (1,0,0), (0,1,0)$ do not span $\mathbb{R}^3$ and are linearly dependent.

7. 2.4.20 - Give an example where uniqueness of representation as for bases fails for linearly dependent sets of vectors. For example $(1,1,0), (1,0,0), (0,1,0)$ are linearly dependent and $(2,2,0) = 2(1,1,0) + 0(1,0,0) + 0(0,1,0)$ and $(2,2,0) = 0(1,0,0) + 2(1,0,0) + 2(0,1,0)$.

8. 2.5.42 - True or false: If $ker(A) = ker(B)$, then $rank(A) = rank(B)$. True. Since $ker(A) = ker(B)$,A and B must have the same number n of columns. Then since $n - rank(B) = dim(ker(B)) = dim(ker(A)) = n - rank(A)$ so the ranks are the same.