Math 242 fall 2008 notes on problem session for week of 10-13-08
This is a short overview of problems that we covered.

1. Prove that if $U \subseteq V$ are vector spaces with $dim(U) = dim(V) = n$ then $U = V$.

   We recall the following. If $dim(V) = n$ and T is a set of n linearly independent vectors in V then T is a basis for V.

   Let T be a basis for U. Since $U \subseteq V$ and $dim(U) = n$, T is a set of n linearly independent vectors in V with $dim(V) = n$ and hence T is a basis for V.

2. Find the projection of $(1, 2, 3)$ onto the line $(1, 1, 1)$. The projection will be a multiple of $(1, 1, 1)$, say $x(1, 1, 1) = (x, x, x)$ such that $(1, 2, 3) - (x, x, x) = (1 - x, 2 - x, 3 - x)$ is orthogonal to the line direction $(1, 1, 1)$. That is $0 = (1 - x, 2 - x, 3 - x) \cdot (1, 1, 1) = (1 - x) + (2 - x) + (3 - x) = 6 - 3x$. So $x = 2$ and the projection is the point $(2, 2, 2)$.

   Do as above with a generic $\boldsymbol{b}$ projected onto a line in the direction $\boldsymbol{a}$ in some $\mathbb{R}^n$. Points on the line are of the form $x\boldsymbol{a}$ for scalars x. We have $(\boldsymbol{b} - x\boldsymbol{a})$ orthogonal to $\boldsymbol{a}$. That is $0 = (\boldsymbol{b} - x\boldsymbol{a}) \cdot \boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{a}x - \boldsymbol{a} \cdot \boldsymbol{a}$ Solving for x we get $x = \frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}$ and the projection is $\boldsymbol{p} = \frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$.

   The square of distance from $\boldsymbol{b}$ to $\boldsymbol{p}$ is nonnegative. Writing this inequality and rearranging we get
   $0 \le (\boldsymbol{b} - \frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}) \cdot (\boldsymbol{b} - \frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}) = \boldsymbol{b} \cdot \boldsymbol{b} - 2\frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{b} \cdot \boldsymbol{a} + (\frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}})^2\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{b} - \frac{(\boldsymbol{b}\cdot\boldsymbol{a})^2}{\boldsymbol{a}\cdot\boldsymbol{a}}$. So $0 \le \boldsymbol{b} \cdot \boldsymbol{b} - \frac{(\boldsymbol{b}\cdot\boldsymbol{a})^2}{\boldsymbol{a}\cdot\boldsymbol{a}}$ and hence $\frac{(\boldsymbol{b}\cdot\boldsymbol{a})^2}{\boldsymbol{a}\cdot\boldsymbol{a}} \le \boldsymbol{b} \cdot \boldsymbol{b}$ which is $(\boldsymbol{b} \cdot \boldsymbol{a})^2 \le (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b})$, the Cauchy-Schwarz inequality for $\mathbb{R}^n$. Here we have assumed $\boldsymbol{a} \cdot \boldsymbol{a} > 0$. The inequality for $\boldsymbol{a} = \boldsymbol{0}$ is trivial.

3. Rewrite the inequality $\left\| \boldsymbol{x} \left\| \boldsymbol{y} \right\| - \boldsymbol{y} \left\| \boldsymbol{x} \right\| \right\|^2 \ge 0$ to obtain the Cauchy-Schwarz inequality. Note first that we can assume that $\|\boldsymbol{x}\| > 0$ and $\|\boldsymbol{y}\| > 0$ as the inequality holds trivially if $\boldsymbol{x} = \boldsymbol{0}$ or $\boldsymbol{y} = \boldsymbol{0}$.

   Using bilinearity and symmetry of inner products and $\langle \boldsymbol{z}, \boldsymbol{z} \rangle = \|\boldsymbol{z}\|^2$ we get
   $0 \le \left\| \boldsymbol{x} \left\| \boldsymbol{y} \right\| - \boldsymbol{y} \left\| \boldsymbol{x} \right\| \right\|^2 = \langle \boldsymbol{x} \left\| \boldsymbol{y} \right\| - \boldsymbol{y} \left\| \boldsymbol{x} \right\|, \boldsymbol{x} \left\| \boldsymbol{y} \right\| - \boldsymbol{y} \left\| \boldsymbol{x} \right\| \rangle = \|\boldsymbol{y}\|^2 \langle \boldsymbol{x}, \boldsymbol{x} \rangle - 2\|\boldsymbol{x}\| \, \|\boldsymbol{y}\| \, \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \|\boldsymbol{x}\|^2 \langle \boldsymbol{y}, \boldsymbol{y} \rangle = 2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2 - 2\|\boldsymbol{x}\| \, \|\boldsymbol{y}\| \, \langle \boldsymbol{x}, \boldsymbol{y} \rangle$. Dividing by $2\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$ and rearranging this becomes $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \le \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$.

4. Show $(a_1 + a_2 + \cdots + a_n)^2 \le n(a_1^2 + a_2^2 + \cdots + a_n^2)$ for real numbers n. This follows immediately by substituting $x_i = 1$ and $y_i = a_i$ for $i = 1, 2, \ldots, n$ into the Cauchy-Schwarz inequality: $(x_1y_1 + x_2y_2 + \cdots + x_ny_n) \le (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2)$.