Right inverse implies left inverse and vice versa
Notes for Math 242, Linear Algebra, Lehigh University fall 2008

These notes review results related to showing that if a square matrix A has a right inverse then it has a left inverse and vice versa. We begin by reviewing the result from the text that for square matrices A we have that A is nonsingular if and only if $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for all $\boldsymbol{b}$.

Nonsingular if and only if unique solutions

Recall that for a square matrix A with factorization $PA = LU$ we have defined A to be nonsingular if the diagonal entries of U are all nonzero and it is singular otherwise. Technically the definition was if there was some such factorization (and it was stated in the text in terms of Gaussian elimination). In fact it doesn't matter which factorization we choose, they will all give the same conclusion. This will follow from the next result that a square matrix A has some such factorization (A is nonsingular) if and only if and only if $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for all $\boldsymbol{b}$.

For square A and $PA = LU$ with U having nonzero diagonal entries, $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for all $\boldsymbol{b}$.
In each factorization $PA = LU$ we have P a permutation matrix and L lower triangular with 1's on the diagonal and U is upper triangular. If U has nonzero diagonal entries then when we solve $U\boldsymbol{x} = \boldsymbol{c}$ for any $\boldsymbol{c}$ we get a unique solution using back substitution. So in this case we solve $A\boldsymbol{x} = \boldsymbol{b}$ by solving $L\boldsymbol{c} = P\boldsymbol{b}$ for $\boldsymbol{c}$ by forward substitution. This always has a solution as the diagonals are nonzero. Then we solve $U\boldsymbol{x} = \boldsymbol{c}$ for any $\boldsymbol{c}$. So if U has nonzero diagonal entries then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for all $\boldsymbol{b}$.

For an $m \times m$ matrix A and $PA = LU$, if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all $\boldsymbol{b}$ then U has nonzero diagonal entries.
For the converse we are showing something slightly stronger. We do this by proving the (equivalent) contrapositive: If 'Not' (U has nonzero diagonal entries) then 'Not' ($A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all $\boldsymbol{b}$). This is: If U has some diagonal entry equal to 0 then for some $\boldsymbol{b}$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution. If some diagonal entry of U is 0 then some column is not a pivot column and hence (because U is square) some row is not a pivot row. Some row of U has all 0's, In particular the last row does. Consider e_m, the vector with every entry 0 except the m^{th} which is 1. Let $\boldsymbol{b} = P^{-1}L\boldsymbol{e_m}$ (since P is a permutation matrix P^{-1} exists and $P^{-1} = P^T$). Then we have $L\boldsymbol{e_m} = P\boldsymbol{b}$. Attempting to solve $U\boldsymbol{x} = \boldsymbol{e_m}$ using back substitution last row of the system $U\boldsymbol{x} = \boldsymbol{c}$ becomes $0 = 1$. So there is no solution for this $\boldsymbol{b}$.

What we have now is A nonsingular $\Rightarrow$ $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for all $\boldsymbol{b} \Rightarrow A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all $\boldsymbol{b} \Rightarrow A$ nonsingular. So these are all equivalent. Note this is only true for square matrices. We also have $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all $\boldsymbol{b} \Leftrightarrow A$ has a right inverse. It remains to establish the A has a left inverse if and only if it has a right inverse.

Left inverse if and only if right inverse

We now want to use the results above about solutions to $A\boldsymbol{x} = \boldsymbol{b}$ to show that a square matrix A has a left inverse if and only if it has a right inverse. Recall also that this gives a unique inverse. Note, this statement is *not* true for non-square matrices. We will assume that A is square.

For a square matrix A, $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all $\boldsymbol{b}$ if and only if A has a right inverse.
If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all $\boldsymbol{b}$ then in particular it does for e_i, $i = 1, 2, \ldots, n$ which are columns of an identity matrix. Then the matrix with i^{th} column equal to the solution of $A\boldsymbol{x} = e_i$ is a right inverse of A. Conversely, if A has a right inverse, Y (such that $AY = I$) then given $\boldsymbol{b}$, the vector Xb solves $A\boldsymbol{x} = \boldsymbol{b}$ since $A(Y\boldsymbol{x}\boldsymbol{b}) = (AY)\boldsymbol{b} = I\boldsymbol{b} = \boldsymbol{b}$.

False Proof
First we give an *incorrect* 'proof' that A has a left inverse implies A has a right inverse. If $XA = I$, consider $A\boldsymbol{x} = \boldsymbol{b}$. Left multiply this by X to get $XA\boldsymbol{x} = X\boldsymbol{b}$ which is $\boldsymbol{x} = X\boldsymbol{b}$. So $A\boldsymbol{x} = \boldsymbol{b}$ always has a solution and by the results above A has a right inverse. *This is not correct.* Why? If $A\boldsymbol{x} = \boldsymbol{b}$ has no solution then the implications that we get be left multiplying are not necessarily true.

Instead we will show first that A has a right inverse implies that A has a left inverse. Then we use this fact to prove that left inverse implies right inverse.

If a square matrix A has a right inverse then it has a left inverse.
Assume that A has a right inverse. From above, A has a factorization $PA = LU$ with L lower triangular with ones on the diagonal and U upper triangular with nonzero diagonal entries. Because the diagonal entries of these triangular matrices are nonzero we can easily see that by forward or back substitution we can solve $\boldsymbol{y}^T L = \boldsymbol{b}^T$ and $\boldsymbol{y}^T U = \boldsymbol{b}^T$ (uniquely) for any $\boldsymbol{b}$. In particular, we can solve for for e_i^T, $i = 1, 2, \ldots, n$ which are rows of an identity matrix. Then the matrix with i^{th} row equal to the solution of $\boldsymbol{y}^T L = \boldsymbol{e_i}^T$ is a left inverse of L. Similarly U has a left inverse. So we have left inverses $\hat{L}$ and $\hat{U}$ with $\hat{L}L = I$ and $\hat{U}U = I$. Now, $(\hat{U}\hat{L}P)A = \hat{U}\hat{L}LU = \hat{U}U = I$. So $\hat{U}\hat{L}P$ is a left inverse of A.

If a square matrix A has a left inverse then it has a right inverse.
Assume that$A$ has a left inverse X such that $XA = I$. Now $A^T X^T = (XA)^T = I^T = I$ so X^T is a right inverse of A^T. By the previous paragraph X^T is a left inverse of A^T. Thus $AX = (X^T A^T)^T = I^T = I$. So A has a right inverse.

Uniqueness of inverses.
Finally we will review the proof from the text of uniqueness of inverses. We note that in fact the proof shows that if X is a left inverse of A and Y is a right inverse of A then $X = Y$. We do not need the more general assumption that X and Y are inverse on both sides. The proof becomes: If $XA = I$ and $AY = I$ then by associativity $X = XI = X(AY) = (XA)Y = IY = Y$.