Math 242 Exam 3 Lehigh University 11-24-2008

This is closed book, closed notes etc. You have 50 minutes to take this exam.

Points for each problem are indicated as $[\cdot]$.

1: [8] Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be elements of an inner product space. Prove that $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2+\|\boldsymbol{w}\|^2$ if and only if $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal.

2: [7] Prove that the transpose of an orthogonal matrix is orthogonal.

3: [17] Let $P = A(A^TA)^{-1}A^T$ be the projection matrix onto the column space of A.
(a) Prove that $P^2 = P$.
(b) Prove that $Range(P) = Range(A)$.

4: [11] Prove that if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are nonzero mutually orthogonal vectors in a vector space then they are linearly independent.

5: [22] Let K be a symmetric matrix and $\boldsymbol{x}^*$ a solution to $K\boldsymbol{x} = \boldsymbol{f}$.
Show/explain each of the following:
(a) The quadratic form $p(\boldsymbol{x}) = \boldsymbol{x}^TK\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{f} + c$ is equal to
$(\boldsymbol{x} - \boldsymbol{x}^*)^TK(\boldsymbol{x} - \boldsymbol{x}^*) + [c - (\boldsymbol{x}^*)^TK\boldsymbol{x}^*]$.
(b) If K is positive definite then $\boldsymbol{x}^*$ exists and is the unique minimizer to $p(\boldsymbol{x})$.
(c) If K is positive semidefinite and $\boldsymbol{x}^*$ exists (i.e., $\boldsymbol{f}$ is in the range of K) then $\boldsymbol{x}$ such that $K\boldsymbol{x} = \boldsymbol{f}$ minimize $p(\boldsymbol{x})$ and the minimizer is not unique.
(d) If K is positive semidefinite and $K\boldsymbol{x} = \boldsymbol{f}$ has no solution (i.e., $\boldsymbol{f}$ is not in the range of K) then $p(\boldsymbol{x})$ has no global minimum.
(e) If K is not positive semidefinite then $p(\boldsymbol{x})$ has no global minimum.

6: [11] Derive the normal equations for the least squares solution to the system $A\boldsymbol{x} = \boldsymbol{b}$. A least squares solution minimizes $\|A\boldsymbol{x} - \boldsymbol{b}\|$ and the orthogonal projection is the vector $\boldsymbol{w} \in W$ such that $\boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}$ is orthogonal to every vector in W. You may assume the geometry that the closest point $\boldsymbol{b}$ in a subspace W to $\boldsymbol{v}$ is the orthogonal projection of $\boldsymbol{b}$ onto W.

7: [24] (a) Find the projection of $(15, -5, 0)^T$ onto $(3/5, 4/5, 0)^T$.
(b) Find the projection of $(5, -10, 2)^T$ onto the plane spanned by $(3/5, 4/5, 0)^T$ and $(4/5, -3/5, 0)^T$.
(c) Find a QR factorization of $\begin{pmatrix} 6 & 15 & 5 \\ 8 & -5 & -10 \\ 0 & 0 & 2 \end{pmatrix}$.