Math 242 Exam 2 Lehigh University 10-29-2008
This is closed book, closed notes etc. You have 50 minutes to take this exam.
Points for each problem are indicated as $[\cdot]$.

**1:** [9] Prove that if $U \subseteq V$ are vector spaces with $dim(U) = dim(V) = n$ then $U = V$.

**2:** [9] For vectors in $\mathbb{R}^2$, $\boldsymbol{v} = (v_1, v_2)^T$ and $\boldsymbol{w} = (w_1, w_2)^T$ define $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = v_1w_1 - v_1w_2 - v_2w_1 + bv_2w_2$. One can check that this satisfies the bilinearity and symmetry conditions for an inner product. Prove that this satisfies positivity: $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ whenever $\boldsymbol{v} \neq \mathbf{0}$ while $\langle \mathbf{0}, \mathbf{0} \rangle = 0$ if and only if $b > 1$. Note, you do not need to check bilinearity and symmetry.

**3:** [9] Prove that for matrices $A, B$, if $BA$ is defined then $ker(A) \subseteq ker(BA)$.

**4:** [13] Answer *one* of the following.
(a) Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis for $\mathbb{R}^n$ and that $A$ is a nonsingular matrix. Prove that $A\boldsymbol{v}_1, A\boldsymbol{v}_2, \ldots, A\boldsymbol{v}_n$ also form a basis for $\mathbb{R}^n$.
(b) Prove that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis for a vector space $V$ if and only if every vector $\boldsymbol{x} \in V$ can be written uniquely as a linear combination of the basis elements.

**5:** [13] Prove the Cauchy-Schwarz inequality $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$.

**6:** [10] Use the Cauchy-Schwarz inequality to prove the triangle inequality $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

**7:** [25] Let $A = LU$ with

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 4 & 4 & 4 & 5 & 7 \\ -1 & -1 & 1 & -2 & -2 & 2 & 4 \\ 0 & 0 & -4 & 0 & 0 & 7 & 3 \\ 3 & 3 & 5 & 6 & 6 & 13 & 15 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 3 & 1 & 0 \\ 3 & 1 & 2 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 1 & 0 & 0 \\ -13 & 5 & -3 & 1 & 0 \\ -7 & 1 & -2 & 0 & 1 \end{bmatrix}.$$

(a) Find a basis for each of the four fundamental subspaces; the kernel, cokernel, range and corange (i.e., nullspace, left nullspace, column space, row space).
(b) Let $S$ be the set of vectors in $\mathbb{R}^5$ consisting of the 7 columns of $A$. Does $S$ span $\mathbb{R}^5$? Is $S$ linearly independent? Explain why in each case.
(c) Pick two of the subspaces in part (a) and give a brief explanation of the general method for finding this basis and why the method works.

**8:** [12] Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. Let $A$ be the matrix with $i^{th}$ column $\boldsymbol{v}_i$ for $i = 1, 2, \ldots, k$ and let $A = LU$ be an $LU$ decomposition of $A$. Let $r$ be the rank of $A$. This is the number of nonzero rows in $U$. For the following you do not need a formal proof. Just give an explanation in terms of solving systems of equations.
(a) Explain why $S$ must be linearly dependent when $k > n$
(b) Explain why $S$ cannot span $\mathbb{R}^n$ when $k < n$.
(c) Explain why $S$ is linearly independent if and only if $S$ spans $\mathbb{R}^n$ when $k = n$.