Notes for Math 242 - Linear Algebra fall 2000 Lehigh University

These are some notes to quickly review material from chapter 2 and add some additional information. They assume familiarity with mmaterial from Math 205 and some of the terminology from basic linear algebra.

1 Matrix multiplication

For various reasons (including applications and the course instructor made me do it) we look at systems of linear equations. For example

$$\begin{array}{ccccccc} 3x & + & 2y & - & z & = & 5 \\ x & + & y & + & 7z & = & 10. \end{array}$$

We would like to write these compactly and we can use this as a motivation for the way matrix multiplication is defined. We write this as $A\mathbf{x} = \mathbf{b}$ where A is the matrix $\begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 7 \end{pmatrix}$, $\mathbf{x}$ is the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{b}$ is the vector $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$. So we have

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

This can also be viewed as a linear combination of 3 vectors in 2-dimensional space

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} x + \begin{pmatrix} 2 \\ 1 \end{pmatrix} y + \begin{pmatrix} -1 \\ 7 \end{pmatrix} z = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

or as the intersection of 2 planes in 3 dimensional space

$$\begin{pmatrix} \begin{pmatrix} 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

In general the system of m equations in the unknowns $x_1, x_2, \ldots, x_m$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

is written in matrix form as $A\mathbf{x} = \mathbf{b}$. That is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

This can also be viewed as a linear combination of n vectors in m-dimensional space

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \cdots + \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{pmatrix} x_k + \cdots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or as the intersection of m hyperplanes in n dimensional space

$$\begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix} \cdot \mathbf{x} \\ \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \cdot \mathbf{x} \\ \vdots \\ \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \mathbf{x} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The examples above motivate matrix multiplication in the general case where A is an $m \times n$ matrix and B is a $n \times p$ matrix. Each of the examples and a combination of the examples will give our first three views of matrix multiplication and then we will give a fourth view.

Matrix multiplication: view 1

If A is an $m \times n$ matrix with entries $(A)_{ij}$ and B is a $n \times p$ matrix with entries $(B)_{ij}$ then AB is the $m \times p$ matrix with entries $(AB)_{ij}$ given by

$$(AB)_{ij} = \sum_{k=1}^{n} (A)_{ik}(B)_{kj}.$$

In words: the row i column j entry of the product AB is the (dot/inner) product of row i of A with column j of B.

For example: In

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 5 & -3 & -4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ -3 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -23 & 4 \\ 9 & -7 \end{pmatrix}$$

the entry 4 in row 1 column 2 is obtained from the product

$$\begin{pmatrix} 1 & 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -4 \\ 3 \end{pmatrix} = (1)(2) + (2)(4) + (4)(-4) + (3)(3) = 4.$$

In

$$\begin{pmatrix} 3 & 7 \\ 4 & -1 \\ -6 & 3 \\ 8 & -5 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 & 6 \\ 2 & 4 & -7 & -5 \end{pmatrix} = \begin{pmatrix} 20 & 25 & -40 & -17 \\ 6 & -8 & 19 & 29 \\ -6 & 18 & -39 & -51 \\ 6 & -28 & -11 & 73 \end{pmatrix}$$

the entry 18 in row 3 column 2 is obtained from the product $\begin{pmatrix} -6 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

Matrix multiplication: view 2

If A is an $m \times n$ matrix with columns $col_j(A)$ and B is a $n \times p$ matrix with entries $(B)_{ij}$ then AB is the $m \times p$ matrix with columns $col_j(AB)$ given by

$$Col_j(AB) = \sum_{k=1}^{n} Col_k(A)(B)_{kj}.$$

In words: the j^{th} column of AB equals the product of A times the j^{th} column of B. That is, it is a linear combination of columns of A with multipliers specified by the j^{th} column of B.

For example: In

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 5 & -3 & -4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 4 \\ -3 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -23 & 4 \\ 9 & -7 \end{pmatrix}$$

the second colum of the product is

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} 3 + \begin{pmatrix} 2 \\ -3 \end{pmatrix} 4 + \begin{pmatrix} 4 \\ -4 \end{pmatrix} - 4 + \begin{pmatrix} 3 \\ 2 \end{pmatrix} 3 = \begin{pmatrix} 4 \\ -7 \end{pmatrix}.$$

In

$$\begin{pmatrix} 3 & 7 \\ 4 & -1 \\ -6 & 3 \\ 8 & -5 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 & 6 \\ 2 & 4 & -7 & -5 \end{pmatrix} = \begin{pmatrix} 20 & 25 & -40 & -17 \\ 6 & -8 & 19 & 29 \\ -6 & 18 & -39 & -51 \\ 6 & -28 & -11 & 73 \end{pmatrix}$$

the third column in the product is $\begin{pmatrix} 3 \\ 4 \\ -6 \\ 8 \end{pmatrix} 3 + \begin{pmatrix} 7 \\ -1 \\ 3 \\ 5 \end{pmatrix} - 7 = \begin{pmatrix} -40 \\ 19 \\ -39 \\ -11 \end{pmatrix}$.

Matrix multiplication: view 3

If A is an $m \times n$ matrix with entries $(A)_{ij}$ and B is a $n \times p$ matrix with rows $Row_j(B)$ then AB is the $m \times p$ matrix with rows $Row_j(AB)$ given by

$$Row_i(AB) = \sum_{k=1}^{n} (A)_{ik} Row_k(B).$$

In words: the i^{th} row of AB equals the product of the i^{th} row of A times B. That is, it is a linear combination of rows of B with multipliers specified by the i^{th} row of A.

For example: In

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 5 & -3 & -4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ -3 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -23 & 4 \\ 9 & -7 \end{pmatrix}$$

the second row of the product is
$5 \begin{pmatrix} 2 & 3 \end{pmatrix} + -3 \begin{pmatrix} 1 & 4 \end{pmatrix} + -4 \begin{pmatrix} -3 & -4 \end{pmatrix} + 2 \begin{pmatrix} -5 & 3 \end{pmatrix} = \begin{pmatrix} 9 & -7 \end{pmatrix}$.
In

$$\begin{pmatrix} 3 & 7 \\ 4 & -1 \\ -6 & 3 \\ 8 & -5 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 & 6 \\ 2 & 4 & -7 & -5 \end{pmatrix} = \begin{pmatrix} 20 & 25 & -40 & -17 \\ 6 & -8 & 19 & 29 \\ -6 & 18 & -39 & -51 \\ 6 & -28 & -11 & 73 \end{pmatrix}$$

the third row in the product is
$3 \begin{pmatrix} 2 & -1 & 3 & 6 \end{pmatrix} + 7 \begin{pmatrix} 2 & 4 & -7 & -5 \end{pmatrix} = \begin{pmatrix} -6 & 18 & -39 & -51 \end{pmatrix}$.

Matrix multiplication: view 4

If A is an $m \times n$ matrix with rows $Row_i(A)$ and B is a $n \times p$ matrix with columns $Col_j(B)$ then AB is the $m \times p$ matrix given by

$$AB = \sum_{k=1}^{n} Col_k(A) Row_k(B).$$

In words: AB is the sum of outer products of columns of A with rows of B. Recall that outer products are special matrices with rank 1, any two nonzero rows (columns) are multiples of each other

For example: We get

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 5 & -3 & -4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ -3 & -4 \\ -5 & 3 \end{pmatrix} =$$

$$
\begin{aligned}
& \begin{pmatrix} 1 \\ 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} \begin{pmatrix} -3 & -4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \end{pmatrix} \\
= & \begin{pmatrix} 2 & 3 \\ 10 & 15 \end{pmatrix} + \begin{pmatrix} 2 & 8 \\ -3 & -12 \end{pmatrix} + \begin{pmatrix} -12 & -16 \\ 12 & 16 \end{pmatrix} + \begin{pmatrix} -15 & 9 \\ -10 & 6 \end{pmatrix} \\
= & \begin{pmatrix} -23 & 4 \\ 9 & -7 \end{pmatrix}
\end{aligned}
$$

We get

$$
\begin{aligned}
& \begin{pmatrix} 3 & 7 \\ 4 & -1 \\ -6 & 3 \\ 8 & -5 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 & 6 \\ 2 & 4 & -7 & -5 \end{pmatrix} = \\
& \quad \begin{pmatrix} 6 & -3 & 9 & 18 \\ 8 & -4 & 12 & 24 \\ -12 & 6 & -18 & -36 \\ 16 & -8 & 24 & 48 \end{pmatrix} + \begin{pmatrix} 14 & 28 & -49 & -35 \\ -2 & -4 & 7 & 5 \\ 6 & 12 & -21 & -15 \\ -10 & -20 & 35 & 25 \end{pmatrix} \\
& = \begin{pmatrix} 20 & 25 & -40 & -17 \\ 6 & -8 & 19 & 29 \\ -6 & 18 & -39 & -51 \\ 6 & -28 & -11 & 73 \end{pmatrix}
\end{aligned}
$$

We quickly review/state basic properties of matrix multiplication.

- Matrix multiplication is associative: for example $A(BC) = (AB)C$ if the matrix multiplications are defined.
- Matrix multiplication (and addition) satisfy distributive laws: $A(B + C) = AB + AC$ (left distributive) and $(A + B)C = AC + BC$ (right distributive).
- Matrix multiplication does *not* satisfy the commutative property. Only in special cases do we get $AB = BA$. For most examples we get $AB \neq BA$. Indeed even if AB is defined its possible that BA is not.

We quickly review/state some basic properties of matrix inverses.

- A square $n \times n$ matrix A is invertable (nonsingular) if there exists a B (necessarily of the same size) such that $AB = I_n$ and $BA = I_n$.

- If an inverse exists it is unique so we can give a special symbol A^{-1} for the inverse. To see uniqueness (in fact this shows that if left and right inverses exist they are the same; indeed for a square matrix one can show that if the left inverse exists then so does the right and they are equal): If $AB = I_n$ and $BA = I_n$ then $B = BI_n = B(AC) = (BA)C = I_nC = C$.

- $\left(A^{-1}\right)^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$. Note the change in order of A and B in the multiplication on the right. For the first, since $A^{-1}A = I$, A is a right inverse of A^{-1} and by uniqueness we get that A is the inverse of A^{-1} so $\left(A^{-1}\right)^{-1} = A$. For the second $(AB)\left(B^{-1}A^{-1}\right) = A\left(B^{-1}B\right)A^{-1} = AIA^{-1} = AA^{-1} = I$
so $B^{-1}A^{-1}$ is the inverse of AB.

- If A has an inverse we can easily solve $A\mathbf{x} = \mathbf{b}$ by left multiplying by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$ (and so the solution is unique).

- To compute A^{-1} note that the i^{th} column of A^{-1} is the solution to $A\mathbf{x} = \mathbf{e_i}$ where $\mathbf{e_i}$ is the I^{th} column of the (appropriate size) identity matrix.

- Assume A is an $n \times n$ matrix. Then A is invertable $\Leftrightarrow$ Gaussian elimination on A yields n pivots $\Leftrightarrow$ A has a nonzero determinant $\Leftrightarrow$ the eigenvalues of A are nonzero $\Leftrightarrow$ $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as a unique solution.

We quickly review/state some basic properties of matrix transposes.

- The transpose of A, denoted A^T is the matrix with the i^{th} row of A^T equal to the i^{th} column of A (for all i). So if A is $m \times n$ then A^T is $n \times m$. Another view of the transpose: $\left(A^T\right)_{ij} = (A)_{ji}$ for $i = 1, 2, \ldots n$ and $j = 12, \ldots m$.

- $(A + B)^T = A^T + B^T$. (Note the analog for inverses does not hold.) Also $(AB)^T = B^TA^T$. One way to note this second fact is as follows assuming A is $m \times n$ and B is $n \times p$:
$\left((AB)^T\right)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} (A)_j k(B)_{ki} = \sum_{k=1}^{n} \left(A^T\right)_{kj} \left(B^T\right)_{ik} = \sum_{k=1}^{n} \left(B^T\right)_{ik} \left(A^T\right)_{kj} = \left(B^TA^T\right)_{ij}$

- $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$. To see this note $I = I^T = \left(AA^{-1}\right)^T = \left(A^{-1}\right)^T A^T$.

- A (necessarily square) matrix A is symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$.