Solution problem 7 on homework 2, Math 242, Linear algebra Lehigh University fall 2000

We need to show the equivalance of the following:

A: Exactly one of the following holds:

$$\begin{array}{l}\text{(I) } A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \text{ has a solution } \boldsymbol{x} \\ \text{(II) } \boldsymbol{y}A \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0 \text{ has a solution } \boldsymbol{y}\end{array}$$

B: Exactly one of the following holds:

$$\begin{array}{l}\text{(I) } A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \text{ has a solution } \boldsymbol{x} \\ \text{(II) } \boldsymbol{y}A \geq \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0 \text{ has a solution } \boldsymbol{y}\end{array}$$

First we note that for each it is easy to show that at most one of the systems holds.

If both IA and IIA hold then

$$0 = \boldsymbol{0}\boldsymbol{0} \leq (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) \leq \boldsymbol{y}\boldsymbol{b} < 0$$

a contradiction. We have used $\boldsymbol{y} \geq \boldsymbol{0}$ in the second $\leq$.

If both IB and IIB hold then

$$0 = \boldsymbol{0}\boldsymbol{0} \leq (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) = \boldsymbol{y}\boldsymbol{b} < 0$$

a contradiction.

So for the remainder we will seek to show *at least one of the following holds.*

Assume A and show B

Note following equivalent systems for given A and $\boldsymbol{x}$ (with block matrices having apprpriate sizes):

$$\text{(IB)} \quad \begin{array}{c} A\boldsymbol{x} = \boldsymbol{b} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} A\boldsymbol{x} \leq \boldsymbol{b} \\ -A\boldsymbol{x} \leq -\boldsymbol{b} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \begin{bmatrix} A \\ -A \end{bmatrix} \boldsymbol{x} \leq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \text{(IB')}$$

and

$$\text{(IIB)} \quad \begin{array}{c} \boldsymbol{y}A \geq \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} (\boldsymbol{u}-\boldsymbol{v})A \geq \boldsymbol{0} \\ (\boldsymbol{u}-\boldsymbol{v})\boldsymbol{b} < 0 \\ \boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \begin{bmatrix} \boldsymbol{u} & -\boldsymbol{v} \end{bmatrix} \begin{bmatrix} A \\ -A \end{bmatrix} \geq \boldsymbol{0} \\ \begin{bmatrix} \boldsymbol{u} & -\boldsymbol{v} \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} < 0 \\ \begin{bmatrix} \boldsymbol{u} & -\boldsymbol{v} \end{bmatrix} \geq \boldsymbol{0} \end{array} \quad \text{(IIB')} \; .$$

In the second set given $\boldsymbol{y}$ one can easily pick non-negative $\boldsymbol{u}, \boldsymbol{v}$ such that $\boldsymbol{y} = \boldsymbol{u} - \boldsymbol{v}$.

Now, if (IB) fails then (IB') fails and thus by applying version A, (IIB') has a solution (this uses the fact that (IB') is a special case of (IA)) and the alternative to this special case is (IIB')), and thus (IIB) has a soltuion. So at least one of (IB) or (IIB) has a solution.

Assume B and show A

Note following equivalent systems for given A and $\boldsymbol{x}$ (with block matrices having appropriate sizes):

$$\text{(IA)} \quad \begin{array}{c} A\boldsymbol{x} \le \boldsymbol{b} \\ \boldsymbol{x} \ge \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} A\boldsymbol{x} + I\boldsymbol{s} = \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{s} \ge \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} = \boldsymbol{b} \\ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} \ge \boldsymbol{0} \end{array} \quad \text{(IA')}$$

and

$$\text{(IIA)} \quad \begin{array}{c} \boldsymbol{y}A \ge \boldsymbol{0} \\ \boldsymbol{y} \ge \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y}A \ge \boldsymbol{0} \\ \boldsymbol{y}I \ge \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y} \begin{bmatrix} A & I \end{bmatrix} \ge \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \text{(IIA')} \ .$$

Now, if (IA) fails then (IA') fails and thus by applying version B, (IIA') has a solution (this uses the fact that (IA') is a special case of (IB)) and the alternative to this special case is (IIA')), and thus (IIA) has a soltuion. So at least one of (IA) or (IIA) has a solution.