

A classification of tournaments having an acyclic tournament as a minimum feedback arc set

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Abstract

Given a tournament with an acyclic tournament as a feedback arc set we give necessary and sufficient conditions for this feedback arc set to have minimum size.

Keywords: tournament, digraph, feedback arc set, linear ordering problem, integer program

1 Introduction

A *tournament* is a digraph where the underlying undirected graph is complete. A *feedback arc set* of a digraph is a set of arcs that when reversed makes the resulting digraph acyclic. It is well known that the problem of determining if a given feedback arc set in a digraph has minimum size is NP-hard. We give necessary and sufficient conditions for a feedback arc set to be minimum in the case that the digraph is a tournament and the feedback arc set is an acyclic tournament.

Finding minimum feedback arc sets is equivalent to finding rankings of the vertices that minimize the number of inconsistencies, arcs xy with y ranked ahead of x . The set of inconsistencies corresponds to the feedback arc set.

We will show that if a tournament has a ranking with an acyclic tournament as the associated feedback arc set then there is an optimal ranking that will decompose into parts with the ‘defining’ ranking (putting the vertices of the acyclic tournament in reverse order) and parts with one other form of ranking.

We have looked at the problem that motivated this paper: determining the smallest size of a tournament having a given acyclic digraph as a minimum feedback arc set in [2],[4],[5],[6]. When the acyclic digraph is a tournament, bounds for the size of the larger tournament were found using a particular class of integer programming problems. Our results here were initially motivated by the problem of showing that solutions to these integer programming problems would provide exact solutions. We show here that this is the case.

2 Definitions

As noted above a *tournament* is a digraph where the underlying undirected graph is complete, a *feedback arc set* of a digraph is an acyclic set of arcs when reversed makes the resulting digraph acyclic and a *minimum feedback arc set* is a smallest sized feedback arc set. For a digraph D with vertex set $V(D)$ and arc set $A(D)$ we will use V and A when there is no ambiguity. A *ranking* of the vertices V is a bijection between V and $\{1, 2, \dots, |V|\}$.

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Definition 1 Let T be a tournament and let π be a ranking of the vertices of T . The set of *inconsistencies* with respect to π , denoted $INC(\pi, T)$, is the set $\{(x, y) \in A(T) | \pi(x) > \pi(y)\}$. Given a tournament T , a ranking π is said to be *optimal* if for any ranking π' of $V(T)$, $|INC(\pi, T)| \leq |INC(\pi', T)|$.

A ranking is optimal if and only if the corresponding set of inconsistencies forms a minimum feedback arc set.

For $\mathbf{x} = (x_0, x_1, x_2, \dots, x_n)$ we use $T(\mathbf{x}, n)$ to describe a tournament that has the acyclic tournament T_n as a feedback arc set as follows. First define T_n by $V(T_n) = \{v_1, v_2, \dots, v_n\}$ and $A(T_n) = \{(v_j, v_i) | j > i\}$. Then $V(T(\mathbf{x}, n)) = V(T_n) \cup \{u_{i,j} | 0 \leq i \leq n, 1 \leq j \leq x_i\}$ and $A(T(\mathbf{x}, n)) = A(T_n) \cup \{(u_{i,j}, u_{s,t}) : i < s \text{ or } i = s \text{ and } j < t\} \cup \{(v_i, u_{s,t}) | i \leq s\} \cup \{(u_{i,j}, v_s) | i < s\}$. That is, $V(T(\mathbf{x}, n))$ consists of the vertices of T_n along with sets of ‘extra’ vertices $U_i = \{u_{i,j} | 1 \leq j \leq x_i\}$ appearing between v_i and v_{i+1} in the ranking σ which makes T_n a feedback arc set. Thus the arc set consists of those arcs consistent with the ranking

$$\sigma = \langle u_{0,1}, \dots, u_{0,x_0}, v_1, u_{1,1}, \dots, u_{1,x_1}, v_2, \dots, v_{n-1}, u_{n-1,1}, \dots, u_{n-1,x_{n-1}}, v_n, u_{n,1}, \dots, u_{n,x_n} \rangle$$

except for arcs $v_i v_j$, which are inconsistent with the ranking. We will refer to this ranking as the *defining ranking*. As the notation implies, tournaments with acyclic subtournaments as a feedback arc set are determined by the size n of the acyclic tournament and the sizes x_i of the sets of ‘extra’ vertices between vertices of the acyclic tournament in the defining ranking.

We will see that for our purposes we will be able to treat segments of $u_{i,j}$ vertices with the same index i as a group. Thus, for simplicity we denote $\langle u_{i,1}, \dots, u_{i,x_i} \rangle$ using U_i . Note that each vertex in U_i has the same adjacencies to vertices outside of U_i . Thus we will refer to arcs (z, U_i) and (U_i, z) with no ambiguity. Vertices v_1, v_2, \dots, v_n will be referred to as v -vertices and sets U_0, U_1, \dots, U_n will be referred to as U -sets.

In order for T_n to be a minimum feedback arc set of $T(\mathbf{x}, n)$ any ranking must have at least as many inconsistencies as the defining ranking. By looking at one class of alternate rankings we will derive bounds on the sizes x_i . We will then show that it is enough to check only these particular alternate rankings to determine if the defining ranking is optimal for a given \mathbf{x} .

For the *block ranking* we place the v -vertices in the ‘middle’ in ‘correct’ order (with no inconsistencies) and the U -sets in ‘correct’ order at the ends as follows;

$$\langle U_0, U_1, U_2, \dots, U_{\lfloor \frac{n}{2} \rfloor}, v_n, v_{n-1}, \dots, v_1, U_{\lfloor \frac{n}{2} \rfloor + 1}, \dots, U_{n-1}, U_n \rangle$$

So all of the inconsistencies are between v -vertices and U -sets.

A *hybrid ranking* is a ranking that can be partitioned into segments, where each of the parts has the form of either a defining or block ranking. For example

$$\langle U_0, U_1, v_3, v_2, v_1, U_2, U_3, v_4, U_4, v_5, U_5 \rangle.$$

In any optimal ranking U_0 will come first and U_n will come last. Thus since U_0 and U_n can be arbitrary for the purposes of determining if T_n is a minimum feedback arc set we will omit them from consideration in what follows. If $U_i = \emptyset$ for some $1 < i < n$ we will include a U_i (with size 0) for convenience in notation.

If T_n is a minimum feedback arc set of $T(\mathbf{x}, n)$, then $x_i \geq 1$ for $1 \leq i \leq n-1$. Otherwise v_i and v_{i+1} are adjacent in the ranking and switching them yields a ranking with fewer inconsistencies.

For the block ranking, it is easy to see that the only inconsistencies are those between U_i and v_j for $i \leq \lfloor \frac{n}{2} \rfloor$ and $j \leq i$ and for $i \geq \lfloor \frac{n}{2} \rfloor + 1$ and $j < i$. So for example, the block ranking

$\langle U_0, U_1, U_2, v_5, v_4, v_3, v_2, v_1, U_3, U_4, U_5 \rangle$ has $x_1 + 2x_2 + 2x_3 + x_4$ inconsistencies. Since the defining ranking in this example has $\binom{5}{2} = 10$ inconsistencies a necessary condition for the defining ranking to be optimal is that $x_1 + 2x_2 + 2x_3 + x_4 \geq 10$. In the general case we get the inequalities $x_1 + 2x_2 + \dots + \frac{n-1}{2}x_{(n-1)/2} + \frac{n-1}{2}x_{(n-1)/2+1} + \dots + 2x_{n-2} + x_{n-1} \geq \binom{n}{2}$ when n is odd and $x_1 + 2x_2 + \dots + \frac{n}{2}x_{n/2} + (\frac{n}{2} - 1)x_{(n/2+1)} + \dots + 2x_{n-2} + x_{n-1} \geq \binom{n}{2}$ when n is even.

By considering the hybrid ranking obtained by switching the subtournament $\langle v_j, U_j, v_{j+1}, U_{j+1}, \dots, U_{h-1}, v_h \rangle$ to the block ranking we get similar inequalities. Note that if a consecutive segment of a ranking is not optimal then the entire ranking is not. Thus we get the following necessary conditions noted in [2] for $T(\mathbf{x}, n)$ to have T_n as a minimum feedback arc set:

$$\sum_{i=1}^{\frac{h-j}{2}} i(x_{j+i-1} + x_{h-i}) \geq \binom{h-j+1}{2} \text{ for } h-j \text{ even} \quad (1)$$

$$\left(\sum_{i=1}^{\frac{h-j-1}{2}} i(x_{j+i-1} + x_{h-i}) \right) + \frac{h-j+1}{2}x_{j+(h-j+1)/2} \geq \binom{h-j+1}{2} \text{ for } h-j \text{ odd} \quad (2)$$

where the \sum term is interpreted as 0 if $h-j=1$.

We will show that these conditions are also sufficient for T_n to be a minimum feedback arc set.

3 Optimal Rankings

Our aim is to show that if (1) and (2) hold then T_n is a minimum feedback arc set of $T(\mathbf{x}, n)$. This is equivalent to showing that the defining ranking is optimal in these cases.

Given any optimal ranking π of $V(T(\mathbf{x}, n))$ we will show that it can be transformed into a hybrid ranking π' without increasing the number of inconsistencies. So π' is optimal. Then (1) and (2) imply that π' has at least as many inconsistencies as the defining ranking. So the defining ranking must also be optimal.

Definition 2 Let y and z be two v -vertices, u -vertices or U -sets in a ranking π such that $\pi(y) < \pi(z)$.

(I) **Switching** y and z switches the place of y and z in the ranking. That is, switching creates a new ranking π' such that (i): $\pi'(y) = \pi(z)$ and $\pi'(z) = \pi(y)$ and (ii): $\pi'(w) = \pi(w)$ for all other vertices w .

(II) **Moving y to the immediate left of z** places y just before z in the ranking. That is, moving y to the left of z creates a new ranking π' such that (i): $\pi'(y) = \pi(z) - 1$, (ii): $\pi'(u) = \pi(u) - 1$ for $\pi(y) < \pi(u) < \pi(z)$ and (iii): $\pi'(w) = \pi(w)$ for all other vertices w . **Moving z to the immediate right of y** is defined in a similar manner.

We will say that a rearrangement (such as switching or moving) of a ranking π to π' is **neutral** if $|INC(\pi', T)| = |INC(\pi, T)|$, is **positive** if $|INC(\pi', T)| - |INC(\pi, T)| > 0$ and is **negative** if $|INC(\pi', T)| - |INC(\pi, T)| < 0$. Thus a ranking π is not optimal if it admits a negative rearrangement.

Parts of the next lemma can easily be seen to be instances of a more general results.

Lemma 3 (i) If y and z appear consecutively in an optimal ranking π and yz is an arc then $\pi(y) = \pi(z) - 1$.

(ii) If π is an optimal ranking of $T(\mathbf{x}, n)$ then $\pi(u_{i,j}) < \pi(u_{s,t})$ for $i < s$ or $i = s$ and $j < t$.

(iii) There exists an optimal ranking π of $T(\mathbf{x}, n)$ such that for each i the vertices in the set U_i appear consecutively.

(iv) If π is an optimal ranking of $T(\mathbf{x}, n)$ and $\pi(v_{i+1}) < \pi(v_i)$ then moving v_i to the immediate right of v_{i+1} and moving v_{i+1} to the immediate left of v_i are neutral.

Proof. (i) Switching y and z removes the inconsistency zy and does not change any other inconsistencies.

(ii) Assume $\pi(u_{s,t}) < \pi(u_{i,j})$ and that π' is obtained from π by switching $u_{s,t}$ and $u_{i,j}$. This removes the inconsistency $u_{i,j}u_{s,t}$ and does not create any new inconsistencies since in this case if $u_{s,t}x$ is an arc then so is $u_{i,j}x$. The switch is negative, contradicting the optimality of π .

(iii) By (i), $\pi(u_{i,j}) < \pi(u_{i,j+1})$. We will show that if $u_{i,j}$ and $u_{i,j+1}$ are not consecutive in π then $u_{i,j}$ can be moved to the immediate left of $u_{i,j+1}$ and $u_{i,j+1}$ can be moved to the immediate right of $u_{i,j}$. Repeating either of these for $j = 1, 2, \dots$ establishes the result.

For z distinct from $\{u_{i,j}, u_{i,j+1}\}$ we have $u_{i,j}z \in A \Leftrightarrow u_{i,j+1}z \in A$. Let $S^+ = \{x | \pi(u_{i,j}) < \pi(x) < \pi(u_{i,j+1}) \text{ and } u_{i,j}x, u_{i,j+1}x \in A\}$ and let $S^- = \{x | \pi(u_{i,j}) < \pi(x) < \pi(u_{i,j+1}) \text{ and } xu_{i,j}, xu_{i,j+1} \in A\}$. If π' is obtained from π by moving $u_{i,j}$ to the immediate left of $u_{i,j+1}$ then $INC(\pi') - INC(\pi) = S^+ - S^-$. If π'' is obtained from π by moving $u_{i,j+1}$ to the immediate right of $u_{i,j}$ then $INC(\pi'') - INC(\pi) = S^- - S^+$. Since π is optimal neither of these switches can be negative. Noting that S^+ and S^- are disjoint we have $|S^+| - |S^-| \geq 0$ and $|S^-| - |S^+| \geq 0$. Hence $|S^-| = |S^+|$ and both switches are neutral.

(iv) Let $S_{i+1}^+ = \{x | \pi(v_{i+1}) < \pi(x) < \pi(v_i) \text{ and } v_{i+1}x \in A\}$, $S_i^+ = \{x | \pi(v_{i+1}) < \pi(x) < \pi(v_i) \text{ and } v_ix \in A\}$, $S_{i+1}^- = \{x | \pi(v_{i+1}) < \pi(x) < \pi(v_i) \text{ and } xv_{i+1} \in A\}$ and $S_i^- = \{x | \pi(v_{i+1}) < \pi(x) < \pi(v_i) \text{ and } xv_i \in A\}$. If π' is obtained from π by moving v_{i+1} to the immediate left of v_i then $INC(\pi') - INC(\pi) = S_{i+1}^+ - S_{i+1}^-$. If π'' is obtained from π by moving v_i to the immediate right of v_{i+1} then $INC(\pi'') - INC(\pi) = S_i^- - S_i^+$. Since π is optimal neither of these switches can be negative. Noting that S_i^- is disjoint from S_i^+ and S_{i+1}^- is disjoint from S_{i+1}^+ we have $|S_{i+1}^+| - |S_{i+1}^-| \geq 0$ and $|S_i^-| - |S_i^+| \geq 0$. Since also $S_i^+ \supseteq S_{i+1}^+$ and $S_{i+1}^- \supseteq S_i^-$ we have $0 \leq |S_{i+1}^+| - |S_{i+1}^-| \leq |S_i^+| - |S_i^-| \leq 0$. Hence $|S_{i+1}^+| = |S_{i+1}^-|$ and $|S_i^-| = |S_i^+|$ and both switches are neutral. ■

By parts (ii) and (iii) of Lemma 3 there are always optimal rankings π where the $u_{i,j}$ vertices can be treated as U -sets. In addition part (ii) shows that $\pi(U_i) < \pi(U_j)$ for $i < j$. We will assume this is the case in what follows.

Lemma 4 *There exists an optimal ranking of $T(\mathbf{x}, n)$ that is hybrid for each \mathbf{x} and each n .*

Proof. We can ignore U_0 and U_n as these sets appear first and last respectively in any optimal ranking. The proof is by induction on n . The result is trivial when $n = 1$. We will show that there is a ranking with an initial segment that is either defining or block. Then, by induction, there is an optimal hybrid ranking on the remaining vertices. Together these give an optimal hybrid ranking of the tournament, establishing the result.

Note that $(U_j, w) \in A$ except when $w \neq v_1$. Thus at most two vertices can precede U_1 in an optimal ranking and if there are two such vertices one of them must be v_1 . If not, then moving U_1 to the beginning of the ranking would be negative. Then, by Lemma 3(ii) the ranking begins (a) $\langle U_1, \dots \rangle$ or (b) $\langle v_1, U_1, \dots \rangle$ or (c) $\langle v_1, v_j, U_1, \dots \rangle$ or $\langle v_j, v_1, U_1, \dots \rangle$ for some $j \neq 1$.

In case (a) there is some non-empty B that does not contain any U -sets such that π begins $\langle U_1, U_2, \dots, U_k, B, U_{k+1}, \dots \rangle$. Take an optimal ranking of this type with k maximal. By repeated applications of Lemma 3 (iv) we can assume that that $v_j \in B$ implies $v_{j-1} \in B$. Thus the set of vertices in B is $\{v_1, v_2, \dots, v_j\}$ for some j . By repeated applications of Lemma 3 (i) the ranking B is $\langle v_j, v_{j-1}, \dots, v_3, v_2, v_1 \rangle$.

Let π' be obtained from π by moving U_{k+1} to the immediate right of U_k and let π'' be obtained from π by moving U_k to the immediate left of U_{k+1} . By optimality of π and maximality of k these moves are negative and negative or neutral respectively. Thus $INC(\pi') - INC(\pi) = (k+1) - (j - (k+1) + 1) = 2k - j + 2 > 0$ and $INC(\pi'') - INC(\pi) = (j - (k+1) + 1) - k = j - 2k \geq 0$. Hence $2k \leq j \leq 2k + 1$.

So the initial segment of the ranking is a block ranking $\langle U_1, U_2, \dots, U_k, v_j, v_{j-1}, \dots, v_2, v_1, U_{k+1}, U_{k+2}, \dots, U_{j-1}, \dots \rangle$ where $j = 2k$ or $j = 2k + 1$.

For case (b), v_1, U_1 is the defining ranking on these vertices.

For (c), if the ranking begins $\langle v_1, v_j, U_1 \dots \rangle$ then switching v_1 and v_j would be negative contradicting optimality. So the ranking begins $\langle v_j, v_1, U_1 \dots \rangle$ and moving U_1 to first is neutral. The new ranking is an instance of case (a). ■

In some sense the hybrid rankings come close to describing all optimal rankings of $T(\mathbf{x}, n)$. However there are situations where rankings that are not hybrid are also optimal. For example, if the block ranking $\langle U_1, U_2, v_6, v_5, v_4, v_3, v_2, v_1, U_3, U_4, U_5 \rangle$ is optimal then so is the ranking $\langle U_1, U_2, v_6, v_4, v_3, v_2, v_1, U_3, U_4, v_5, U_5 \rangle$ in the case that $|U_3| = |U_4| = 2$.

4 Conditions

Recall that if T_n is a feedback arc set of some larger tournament then the larger tournament is $T(\mathbf{x}, n)$ for some \mathbf{x} . The results of the previous section immediately give our main result showing sufficiency of the necessary conditions on \mathbf{x} for T_n to be a minimum feedback arc set of $T(\mathbf{x}, n)$.

Theorem 5 *If a tournament T has the acyclic tournament T_n on n vertices as a feedback arc set then $T = T(\mathbf{x}, n)$ for some \mathbf{x} and T_n is a minimum feedback arc set of $T(\mathbf{x}, n)$ if and only if*

$$\sum_{i=1}^{\frac{h-j}{2}} i(x_{j+i-1} + x_{h-i}) \geq \binom{h-j+1}{2} \text{ for } h-j \text{ even} \quad (1)$$

$$\left(\sum_{i=1}^{\frac{h-j-1}{2}} i(x_{j+i-1} + x_{h-i}) \right) + \frac{h-j+1}{2} x_{j+(h-j+1)/2} \geq \binom{h-j+1}{2} \text{ for } h-j \text{ odd} \quad (2)$$

where the \sum term is interpreted as 0 if $h-j = 1$.

Proof. We have already noted the necessity of the conditions.

By Lemma 4 there exists a hybrid ranking π which is optimal. We need to show that σ which has T_n as its set of inconsistencies is also optimal. We obtain σ from π by rearranging each segment that is a block ranking into a defining ranking. The conditions (1) and (2) insure that each rearrangement does not increase the number of inconsistencies. Hence σ is also optimal. ■

This result implies a polynomial algorithm for determining if a feedback arc set is minimum in the case that the digraph is a tournament and the feedback arc set is an acyclic tournament. We only need to check the feasibility of $\binom{n}{2}$ inequalities found in the statement of Theorem 5.

The integer programming problems (mentioned in the introduction) used to determine bounds on the size of a smallest tournament having the acyclic tournament of size n as a minimum feedback arc set are $\min \sum_{i=0}^n x_i$ subject to (1) and (2) with the x_i non-negative integers. Theorem 5 shows that exact solutions to these integer programming problems if the solutions could be found in fact give exact bounds to the size problem. An upper bound of $2n - 4$ (for $n \geq 3$) comes from setting $x_0 = x_n = 0$, $x_1 = x_{n-1} = 1$ and all other $x_i = 2$. Feasibility is easily checked. It was also shown directly that for this \mathbf{x} , we have T_n as a minimum feedback arc set of $T(\mathbf{x}, n)$. In [4] a lower bound of $2n - 2 - \lfloor \log_2 n \rfloor$ or $2n - 3 - \lfloor \log_2 n \rfloor$ (depending on the binary expansion of n) is established. In [4] and [1] it is shown that this lower bound is optimal for several infinite classes of values of n . We conjecture that these lower bounds are indeed optimal.

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References

- [1] J. L. Baldwin, W. C. Kronholm, D. A. Narayan, Tournaments with a transitive tournament as a feedback arc set, *Congr. Numer.* 158 (2002) 43-51.
- [2] J.-P. Barthélemy, O. Hudry, G. Isaak, F. S. Roberts, B. Tesman, The reversing number of a digraph, *Disc. Appl. Math.* 60 (1995) 39-76.
- [3] P. Erdos, J. W. Moon, On sets of consistent arcs in a tournament, *Canad. Math. Bull.* 8 (1965) 269-271.
- [4] G. Isaak, Tournaments as feedback arc sets, *Elec. J. Combin.* R20 (1995) 19pp.
- [5] G. Isaak, D. A. Narayan, Complete classification of all tournaments having a disjoint union of directed paths as a minimum feedback arc set, *J. Graph Theory* 45 (2004) 28-47.
- [6] D. A. Narayan, Tournaments with feedback path powers, *J. Combin. Math. Combin. Comput.* 49 (2004) 85-96.
- [7] D. A. Narayan, The reversing number of a digraph; A disjoint union of directed stars, *Congr. Numer.* 145 (2000) 153-164.