# An alternate approach to changing sine

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A review of standard calculus textbooks reveals a common approach to showing that the derivative of the sine function is the cosine function. This approach is to first derive the limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and obtain the corollary $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$ and then use both in the limit definition of derivative, after applying a trigonometric identity for the sine of a sum. While there is some variation in the texts as to how the first limit is obtained using geometry, the final steps using a trigonometric identity and applying both limits is common.

Our goal here is to present an alternative approach for the derivative of sine with the idea that it may provide good motivation for the derivative as a rate of change. In addition we reduce the use of trigonometric identities so that the limit we compute is exactly that for the rate of change in the definition of the derivative. While this approach must be known somewhere, it seems to be hard to track down. The idea here is to advertise it as an approach that provides good motivation to students to view the derivative as a rate of change and to have a simple figure that suggests why the derivative of sine is cosine.

We will observe that the geometry is not much different than approaches used for $\lim_{h \to 0} \frac{\sin h}{h} = 1$, which directly shows that the derivative of sine at 0 is 1 (equal to $\cos 0$). By shifting this along the unit circle we get a simple figure that at least informally motivates the idea that we ought to expect $(\sin x)' = \cos x$.

## Overview

We will proceed as follows with five short sections:

1. Translate the slope of a secant line on $y = \sin x$ to a ratio in a unit circle diagram.

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2. Show directly from geometry on the unit circle the right side limit for angles in the first quadrant.
3. Fill in two easy arc inequalities used in step 2.
4. Discuss how a similar computation yields the left side limit in the first quadrant.
5. Discuss informally how the rate of change ideas extend the derivative to all angles how symmetric arguments would show these formally.

## Rate of change of height along the unit circle

Recalling that the derivative is a rate of change and that values of the sine function are 'heights' ($y$ coordinates) on the unit circle we see that the derivative of sine tells us 'how fast' the 'height' changes as we 'walk' at unit speed along the unit circle.

We start with a look at the translation to the unit circle of a secant line on the graph of $y = \sin x$. Using common calculus notation we consider slope $= \dfrac{\text{rise}}{\text{run}}$ for 'inputs' $x$ and $x + h$ where for now we assume $h$ is positive and $x$ is between 0 and $\frac{\pi}{2}$.

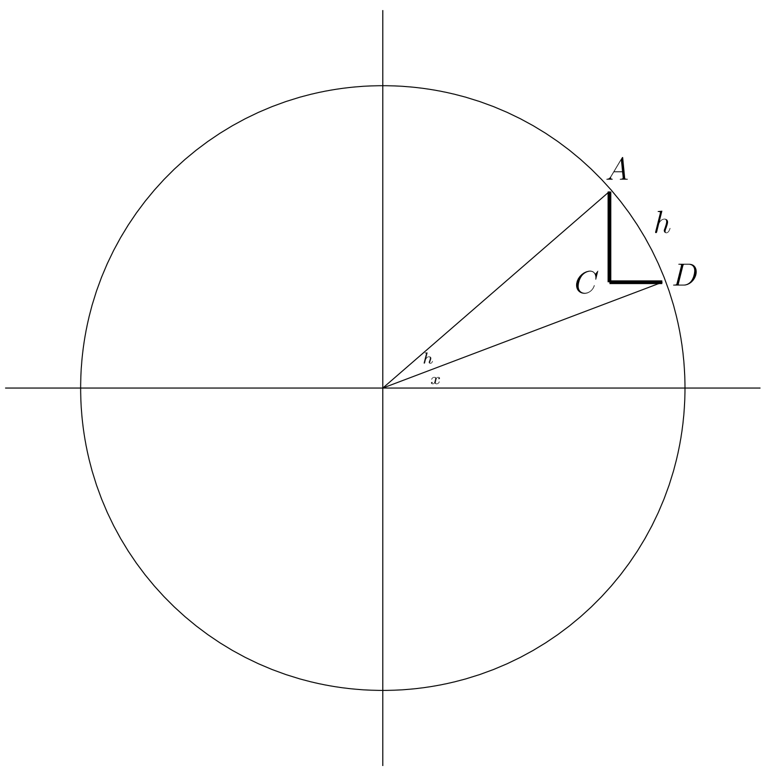


Figure 1

The slope of the secant on the graph $y = \sin x$ is $\dfrac{\sin(x+h) - \sin x}{h}$. Translated to the unit circle this is the ratio of the change in height to the length of the arc. That is, $\frac{|AC|}{h}$ in figure 1. Here we get our first hint that the derivative should be cosine. If we believe that as $h$ gets small the arc $AD$ in the region $ACD$ 'approaches' a straight line so the region 'approaches' a triangle with the angle between the vertical segment and hypotenuse $x+h$ (which approaches $x$), then immediately $\frac{|AC|}{h} = \cos x$ (i.e., the ratio we want approaches opposite over hypotenuse which is cosine).

Of course, this intuition does not provide a proof. For this we bound the slope of the secant using two simple geometric arguments with slightly different triangles.

## Secant limit for positive change $h$ in the first quadrant

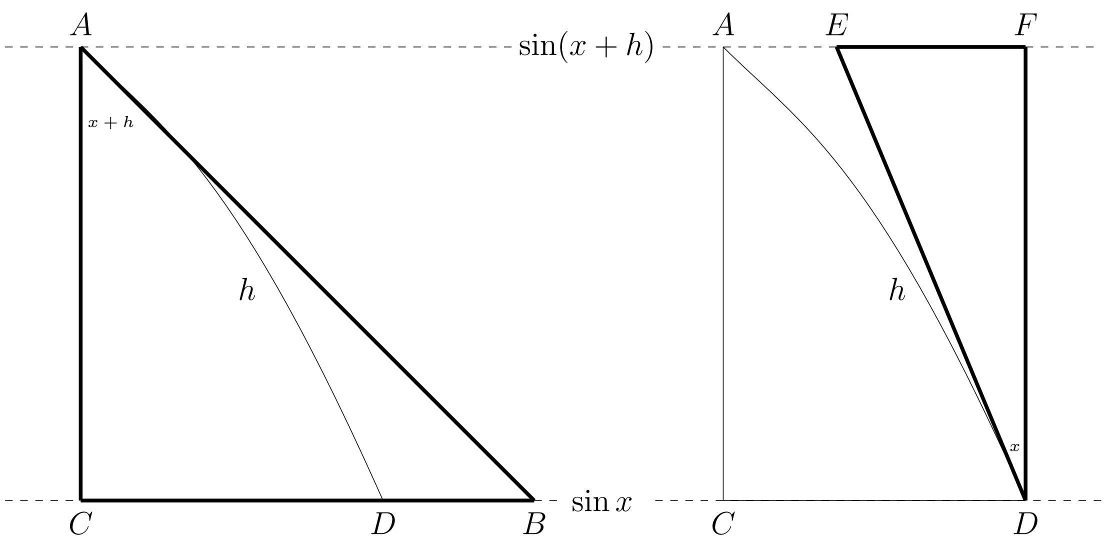


Figure 2

On the left in figure 2 we zoom to the region along the circle add the tangent to the circle at $A$ and its intersection $B$ with the horizontal line at height $\sin x$. Accept for now the intuition that arc $AD$ length $h$ is less than the segment length $|AB|$. We will establish this later. Also observe from basic trigonometry that the angle $\angle BAC$ between the tangent segment $AB$ and the vertical segment $AC$ is $x+h$. Then using $h < |AB|$ and adjacent over hypotenuse in triangle $ABC$ we get

$$\frac{\sin(x+h) - \sin x}{h} > \frac{\sin(x+h) - \sin x}{|AB|} = \frac{|AC|}{|AB|} = \cos(x+h).$$

On the right in figure 2 we zoom to the region along the circle add the tangent to the circle at $D$ and its intersection $E$ with the horizontal line at height $\sin(x+h)$. Accept for now the intuition that arc $AD$ length $h$ is greater than the segment length $|DE|$. We will establish this later. Also observe from basic trigonometry that the angle $\angle EDF$ between the tangent segment $DE$ and the vertical segment $DF$ is $x$. Then using $h > |DE|$ and adjacent over hypotenuse in triangle $DEF$ we get

$$\frac{\sin(x+h) - \sin x}{h} < \frac{\sin(x+h) - \sin x}{|DE|} = \frac{|DF|}{|DE|} = \cos x.$$

Now, for $0 \leq x < \frac{\pi}{2}$ and small positive $h$ we have that $\cos(x+h) < \frac{\sin(x+h)-\sin x}{h} < \cos x$. By the squeeze theorem we get

$$\lim_{h \to 0^+} \frac{\sin(x+h) - \sin x}{h} = \cos x$$

## Arc inequalities

We establish the arc bounds as follows.

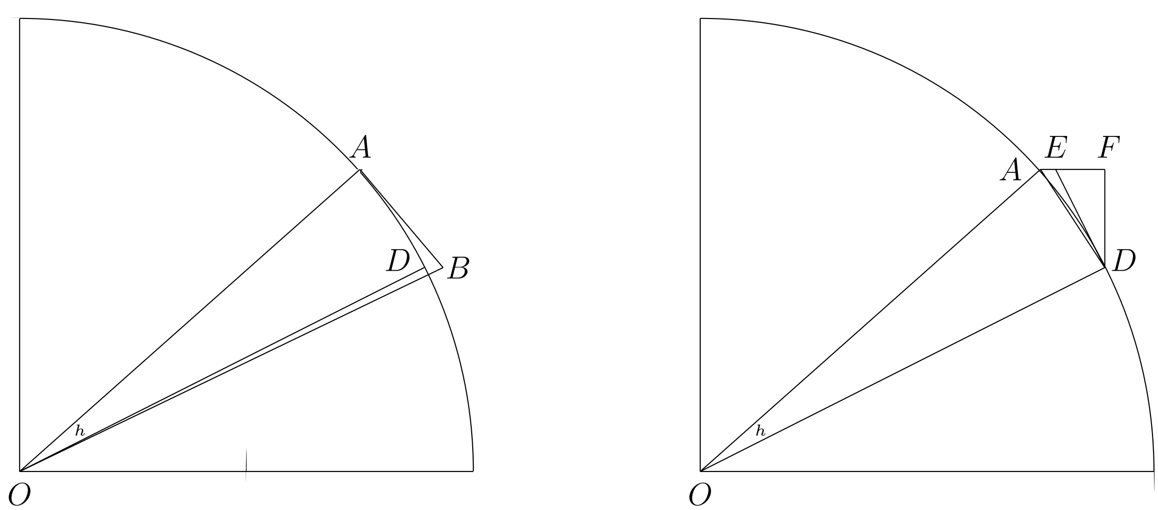


Figure 3

On the left in figure 3, since right triangle $OAB$ contains sector $OAD$ and recalling

$|OA| = 1$ on the unit circle, we get

$$\frac{h}{2} = \frac{h}{2\pi} \cdot \pi \cdot 1^2 = \text{ area of sector } OAD < \text{ area of triangle } OAB = \frac{1}{2} \cdot 1 \cdot |AB| = \frac{|AB|}{2}$$

establishing the first arc inequality $h < |AB|$.

On the right in figure 3, as angle $ADF$ is greater than angle $EDF$, using cosines $\frac{|DF|}{|DA|} < \frac{|DF|}{|DE|}$ from the corresponding cosines. Hence $|DA| > |DE|$. Then since the arc $AD$ length $h$ is greater than the segment length $|DA|$ we get $h > |DA| > |DE|$ establishing the second arc inequality $h > |DE|$.

## Secant limit for negative change $h$

For the left limit consider $h < 0$. The figures are nearly the same as those above for $h > 0$ and are omitted. For the new versions, the top height is $\sin x$, the angle $\angle CAB$ is $x$, the bottom height is $\sin(x + h)$ and angle $\angle FDE$ is $x + h$. In addition the arc length is now $-h$ as $h < 0$. The secant slope is $\dfrac{\sin x - \sin(x + h)}{(-h)} = \dfrac{\sin(x + h) - \sin x}{h}$. From the switching of the angles as described above we end up with $\cos x < \frac{\sin(x+h)-\sin x}{h} < \cos(x + h)$ when $h < 0$. Then from the squeeze theorem we get $\lim\limits_{h\to 0^-} \dfrac{\sin(x + h) - \sin x}{h} = \cos x$ as needed.

## Extending to all inputs

To extend to angles beyond beyond the first quadrant on the unit circle we can easily draw appropriate figures symmetric to those above, make appropriate changes in values and get the limit as $\cos x$. We omit the straightforward details for this.

An alternative informal explanation (not proof) is to note that we have shown that moving along the unit circle at unit speed in the first quadrant the rate of increase of height is the horizontal distance from the vertical axis. In the fourth quadrant we get the same rate of increase and the same horizontal distance $\cos x$. In the 2nd and 3th quadrants, the 'speed' of the rates are the same but the height is decreasing as the distance along the circle increases. Hence the rate of change is the negative of the horizontal distance, which is the $y$ coordinate $\cos x$ as needed.

Note that this informal idea of rate of change of height as we walk with unit speed along the unit circle translates by symmetry to a hint that $(\cos x)' = -\sin x$.