An Edge Count Criterion for Degree Sequences

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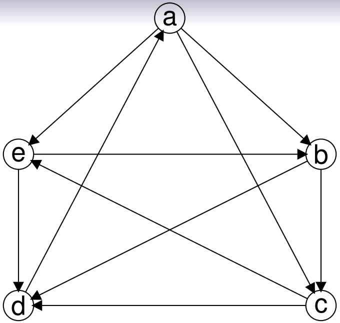
Title: An Edge Count Criterion for Degree Sequences

Subtitle: Degree Sequences for Edge Colored Graphs

Outline:

- (Subtitle) An aside, motivation and history
- (Title) Degree sequences proof
- (Subtitle) Some old and some new results on edge colored degrees sequences



Score sequence: $(3, 2, 2, 2, 1)$

Tournament (complete directed graph)

Score $=$ outdegree (the number of wins)

Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Landau's Theorem: The number of wins for any set of teams must be as large as the number of games played between those teams

and

the total number of wins must equal the total number of games played

Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Landau's Theorem: The number of wins for any set of teams must be as large as the number of games played between those teams

and

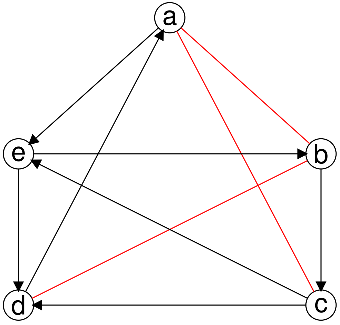
the total number of wins must equal the total number of games played

The sequence is not a score sequence: 9 teams with lowest scores have a total of 44 wins but play $\binom{9}{2} = 45$ games.

What if ties are allowed?

Can we tell if a given set of triples (win,loss,tie) arises from a round robin tournament?

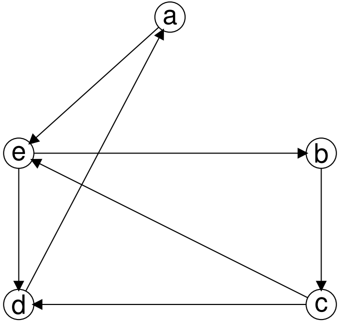
	win	loss	tie
a	1	1	2
b	1	1	2
c	2	1	1
d	1	2	1
e	2	2	0



What if ties are allowed?

Can we tell if a given set of triples (win,loss,tie) arises from a round robin tournament?

	win	loss
a	1	1
b	1	1
c	2	1
d	1	2
e	2	2



Equivalently - Is there a digraph with specified indegrees and outdegrees?

Is there a digraph with specified indegrees and outdegrees?

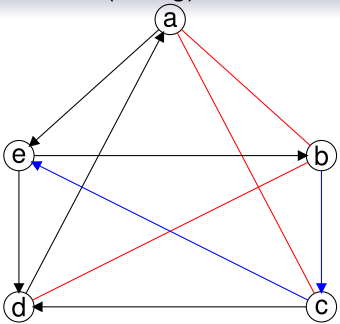
- Gale (1957) and Ryser (1957) give conditions (equivalent to degree sequences of bipartite graphs) - but this allows loops
- Ore (1957) and Fulkerson (1960) give conditions with no loops - but this allows 2-cycles

Surprisingly the (win,loss,tie) version appears to be unsolved

- Consider instead: indegree minus outdegree - conditions given by Avery (1991) for tournaments and Mubayi, West and Will (2001) for simple digraphs

What if we in addition specify shutout (or big) wins?

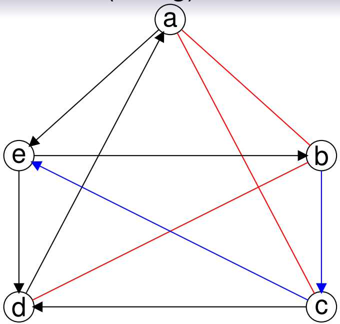
	big win	win	tie	loss	big loss
a	0	1	2	1	0
b	1	0	2	1	0
c	1	1	1	0	1
d	0	1	1	2	0
e	0	2	0	1	1



Can we tell if specified 5-tuples (big win, win, tie, loss, big loss) arise from a round robin tournament?

What if we in addition specify shutout (or big) wins?

	big win	win	tie	loss	big loss
a	0	1	2	1	0
b	1	0	2	1	0
c	1	1	1	0	1
d	0	1	1	2	0
e	0	2	0	1	1



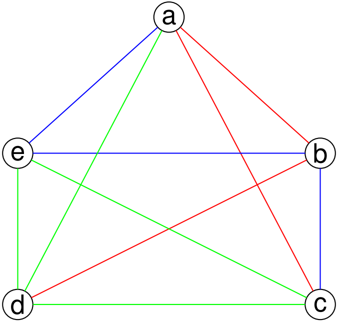
Can we tell if specified 5-tuples (big win, win, tie, loss, big loss) arise from a round robin tournament?

If big win = +2, win = 1, tie = 0, loss = -1, big loss = -2 then can get conditions for total score via Hoffman's circulation theorem

Edge Colored Graphs

Look at (non-proper) edge colorings of complete graphs

	blue	green	red
a	1	1	2
b	2	2	0
c	1	2	1
d	0	3	1
e	2	2	0



Which sequences of vectors can be realized as degrees of an edge colored complete graph?

Columns - color sequences

Rows - color degrees

Which sequences of vectors can be realized as degrees of an edge colored complete graph?

Graphs with (not necessarily proper) edge colorings arise in many contexts

- Distinguishing colorings
- Ramsey theory
- Paths with alternating color
- Etc.

Which sequences of vectors can be realized as degrees of an edge colored complete graph?

- Not really a new question, Rao and Rao (1969) asked:
  - When is a degree sequence realizable by a graph with a k-factor?
- In edge colored graph terminology
  - What are degree sequences of three edge colored graphs if one color sequence is constant?
- Necessary condition: Each color sequence must be realizable as a graph
- Kundu (1973): necessary condition is also sufficient

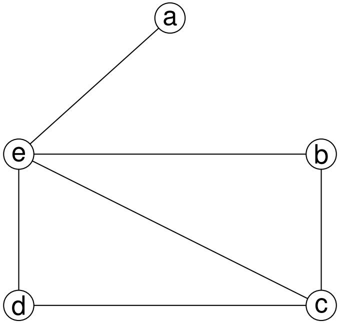
Which sequences of vectors can be realized as degrees of an edge colored complete graph?

- About a dozen papers written on the 3 color version in 1970's and early 1980's
- Can be viewed as a version of graph factorization and of potentially graphic sequences but questions in those settings are usually slightly different
- If the input graph is arbitrary (not complete) then the problem is NP-hard (since: Is a 3 regular graph 3-edge colorable is NP-hard)

k-Edge Colored Graph = Partition of the family of 2 element subsets of V into k classes

Graphs can be viewed as 2 edge colored graphs (partition into edges and non-edges)

Degree Sequence
$4, 3, 2, 2, 1$



Is $8, 8, 8, 7, 4, 4, 3, 2, 1, 0$ a degree sequence?

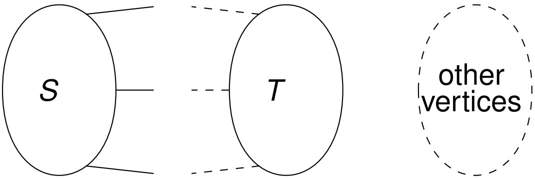
Sierksma and Hoogeveen (1991) review 7 criteria for a graphic sequences:

$d_1 \geq d_2 \geq \cdots \geq d_n$ non-negative integers are graphic if and only if:

- (Erdos-Gallai) $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_i\}$
- (Berge) $\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \overline{d}_i$ where $\overline{d}_i$ is the corrected conjugate sequence.
- (Fulkerson-Hoffman-McAndrew) $\sum_{i=1}^{k} d_i \leq k(n-m-1) + \sum_{i=n-m+1}^{n} d_i$

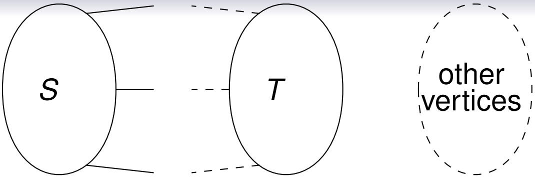
All except Fulkerson-Hoffman-McAndrew involve conjugate sequence or min and/or max

Edge Count Criterion (ECC) for degree sequences



edges out of S $\leq \sum_{i \in S} d_i$

non-edges out of T $\leq \sum_{i \in T} (n - 1 - d_i)$



edges out of S $\le \sum_{i \in S} d_i$

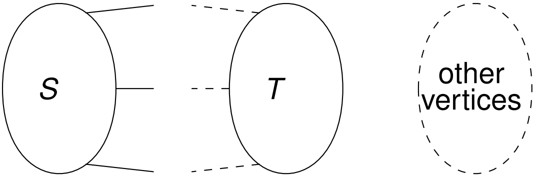
non-edges out of T $\le \sum_{i \in T}(n-1-d_i)$

Necessary Condition
for $(d_1, d_2, \ldots, d_n)$ to be a Degree Sequence
For disjoint S, T (and $\sum d_i$ even)

$$\sum_{i \in S} d_i + \sum_{j \in T}(n-1-d_j) \le |S|\,|T|$$

This is also sufficient

Edge Count Criterion (ECC) for degree sequences



$(d_1, d_2, \ldots, d_n)$ is NOT a degree sequence if
degrees in S plus non-degrees in T
too small to account for edges between S and T

Edge Count Criterion (ECC)

$(d_1, d_2, \ldots, d_n)$ with $\sum d_i$ is a Degree Sequence
if and only if

$$\sum_{i \in S} d_i + \sum_{j \in T} (n - 1 - d_j) \leq |S|\,|T|$$

for all disjoint S, T

- ECC is very close to a degree sequence condition given by Koren (1973).
- Koren's version is used by Peled and Srinivasan (1989) to characterize the polytope of degree sequences

A (new?, simple?) proof for degree sequences:
Use the following simple lemma:
If $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic then there exists a realization with v_n not adjacent to v_j (as long as $d_j \neq n-1$)

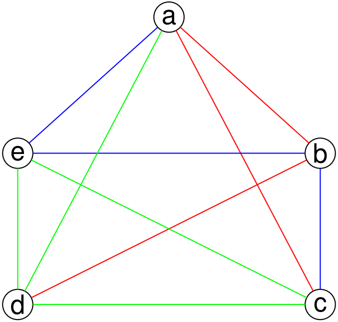
Proof that Edge Count Criterion for $d_1 \geq d_2 \geq \cdots \geq d_n \Rightarrow$ degree sequence
Induction on $n + \sum d_i$:
Show $d_1 - 1, d_2, \ldots, d_{n-1}, d_n$ satisfies ECC and apply lemma to get a realization with v_1, v_n not adjacent. Add that edge.

Edge Count Criterion for bipartite graphs:

Edge Colored Graphs

Look at (non-proper) edge colorings of complete graphs

	blue	green	red
a	1	1	2
b	2	2	0
c	1	2	1
d	0	3	1
e	2	2	0



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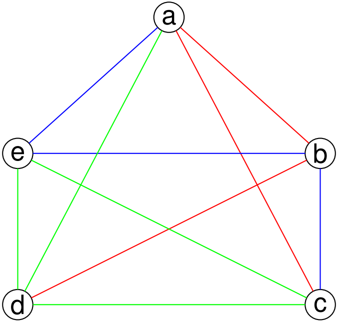


Necessary Condition - each color sequence is a degree sequence

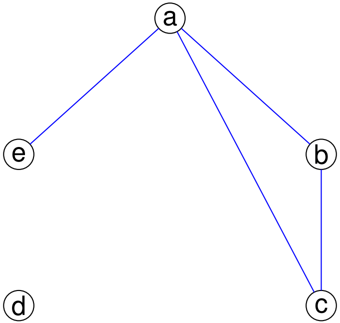
$\geq$ 4 colors: sums of color sequences must be degree sequences

Necessary condition is sufficient (3 colors) when:

- One color sequence has all degrees $\in \{k, k+1\}$ (Kundu's Theorem, 1973)
  - extends to two outlying degrees and ...
- Two color sequences and their sum can be realized by forests (Kleitman, Koren and Li, 1977)
  - also a broader condition on the sum; characterize when both colors can be forests
- Minimum degree for one color is $> (n-1) - \sqrt{2n}$ (Busch and Ferrara, Thursday)

Necessary condition is not sufficient

	blue	green	red
a	3	1	0
b	2	0	2
c	2	0	2
d	1	2	1
e	0	1	3



Necessary condition is not sufficient

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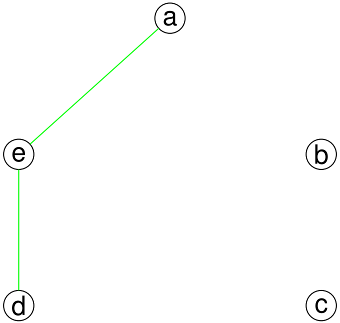

Necessary condition is not sufficient

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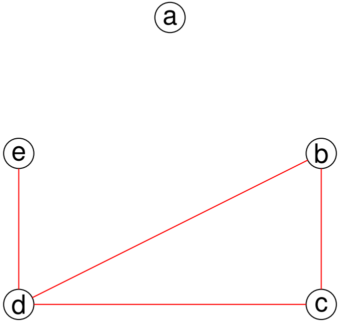


Each color is a degree sequence
Problem: de must be both green and red

Generalization of ECC works to show non realizability in this case.
Does it work always?
Fractional version works for bipartite

If necessary conditions are not sufficient?

- (3 colors) Koren characterized conditions when one color is a star (with possibly isolated vertices)
- (3 colors) If any color vector is uniquely realizable reduces to factor problem in that graph. Uniquely realizable sequences are threshold graphs
- (4 colors) If any sum of 2 colors is uniquely realizable reduces to factor problems in two graphs.

When is there an edge colored tree with given color degrees?

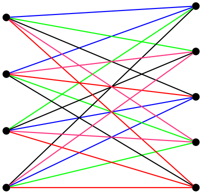
- Trivial if tree is given
- For 2 colors follows from Kleitman, Koren, Li (1973)
- In general each sum of colors must be forest realizable (proof for 3 colors, for more colors probably...)

Is there a bipartite graph with given colors vectors?

$(1, 1, 1, 1, 1)$	•	•	$(2, 1, 1, 0, 0)$
$(1, 1, 1, 1, 1)$	•	•	$(0, 1, 1, 0, 2)$
$(1, 1, 1, 1, 1)$	•	•	$(2, 0, 1, 1, 0)$
$(1, 1, 1, 1, 1)$	•	•	$(0, 2, 0, 0, 2)$
		•	$(0, 0, 1, 3, 0)$

Is there a bipartite graph with given colors vectors?

$(1, 1, 1, 1, 1)$
$(1, 1, 1, 1, 1)$
$(1, 1, 1, 1, 1)$
$(1, 1, 1, 1, 1)$



$(2, 1, 1, 0, 0)$
$(0, 1, 1, 0, 2)$
$(2, 0, 1, 1, 0)$
$(0, 2, 0, 0, 2)$
$(0, 0, 1, 3, 0)$

YES

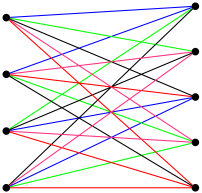
Row sums = column sums; use Birkhoff-VonNeumann Theorem

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$



$(2, 1, 1, 0, 0)$

$(0, 1, 1, 0, 2)$

$(2, 0, 1, 1, 0)$

$(0, 2, 0, 0, 2)$

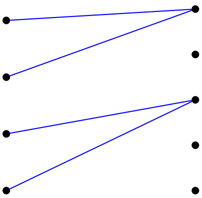
$(0, 0, 1, 3, 0)$

$$\begin{array}{|ccccc|}\hline 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \\ \hline \end{array} = \begin{array}{|ccccc|}\hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline \end{array} + \begin{array}{|ccccc|}\hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline \end{array} +$$

$$\begin{array}{|ccccc|}\hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline \end{array} + \begin{array}{|ccccc|}\hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

This always works when one part degrees are 1's vectors

$(1, 1, 1, 1, 1)$ | $(2, 1, 1, 0, 0)$
$(1, 1, 1, 1, 1)$ | $(0, 1, 1, 0, 2)$
$(1, 1, 1, 1, 1)$ | $(2, 0, 1, 1, 0)$
$(1, 1, 1, 1, 1)$ | $(0, 2, 0, 0, 2)$
 | $(0, 0, 1, 3, 0)$



$$\boxed{\begin{matrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{matrix}} = \boxed{\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}} + \boxed{\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}} +$$

$$\boxed{\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}} + \boxed{\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}}$$

There are some interesting problems with degree sequences of edge colored graphs