Intervals and Orders: What Comes After Interval Orders?

Garth Isaak

Department of Mathematics
Lehigh University

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Special session on the Mathematics of Kenneth P. Bogart

Intervals and Orders: What Comes After Interval Orders?

Kenneth P. Bogart*

Dartmouth College, Hanover, NH, 03755, U.S.A.

Abstract. In this paper we survey two kinds of generalizations of the ideas of interval graphs and interval orders. For the first generalization we use intervals in ordered sets more general than the real numbers. For the second generalization, we restrict ourselves to intervals chosen in the real numbers, but we define two vertices to be adjacent (in the graphs) or incomparable (in the orders) only when the intervals overlap by more than a specified amount. Each of these generalizations suggests new avenues for research.

1 Introduction

Dartmouth College HANOVER • NEW HAMPSHIRE • 03755

Department of Mathematics and Computer Science • Bradley Hall

Kenneth P. Bogart
Professor of Mathematics and Computer Science
Chair

E-Mail K.P.Bogart@Dartmouth.edu
Department Telephone: 603-646-2415
Office Telephone: 603-646-3178

Sept 22, 1989

Garth Isaak
Rutcor
Hill Center
Rutgers University
New Brunswick, NJ
08903

Dear Garth:

I've been wanting to answer your note for a long time, but my new job keeps me busy! I learned a lot from your note and I like the work. I did know about the use of shortest paths and negative cycles in the situation where the interval lengths are specified, but your approach is totally different. Are the digraph you use and the lengths you picked standard techniques or a masterful stroke of insight on your part? Your smooth approach to the problem of determining whether an interval order is representable with intervals of length less than or equal to k has me hoping there is probably a forbidden suborder characterization of these interval orders like the one Karen and I found for semiorders. Any ideas?

If you have an e-mail address, please send it to me by e-mail so I can drop you a note easily.

By the way, the only reason I didn't explicitly say you should write the results up is that I don't know how much of it was standard

From: IN%"Kenneth.P.Bogart@mac.dartmouth.edu" 3-OCT-1989 11:35:04.41
To: ISAAK@cancer.rutgers.edu
CC:
Subj: Re: Interval Order notes

Return-path: Kenneth.P.Bogart@mac.dartmouth.edu
Received: from dartvax.dartmouth.edu by cancer.rutgers.edu; Tue, 3 Oct 89 11:34 EST
Received: from d0.dartmouth.edu by dartvax.dartmouth.edu (5.61D1/4.0HUB) id AA15448; Tue, 3 Oct 89 11:37:47 -0400
Received: by D0.DARTMOUTH.EDU id <4972>; 3 Oct 89 11:37:53
Date: 03 Oct 89 11:37:50
From: Kenneth.P.Bogart@mac.dartmouth.edu
Subject: Re: Interval Order notes
To: ISAAK@cancer.rutgers.edu
Message-Id: <612812@mac.Dartmouth.EDU>

I think the Boca meeting would be a good forum for your ideasnad the results would be appropriate for their proceedings.

I've been waiting for my student to write up the semiorder paper since it is a part of her thesis. I have the paper half-written anyhow from writing lecture notes, so maybe this will propel me to finish off a draft of it! We don't seem to have any rules that would preclude it from appearing in her thesis when she gets around to submitting it.

When you look for jobs, I'd sure be glad to have you look at our postdoctoral research instructorship.

in the sense that $t_\ell(x)$ and $t_r(x)$ are less than or equal to the length of I_x. It is straightforward to show that proper bitolerance orders have a bounded representation.

the center of I_x is to the left of I_y and the length of the overlap of I_x and I_y is less than both $t_r(x)$ and $t_\ell(y)$. As we shall see the bounded bitolerance orders are just orders of interval dimension 2. The assignment of intervals and tolerances to the elements of a (bi)tolerance ordering is called a representation of the ordering.

interval

It is clear that unit (bi)tolerance orders are proper. Thus it is natural to ask whether proper (bi)tolerance orders are unit. Perhaps surprisingly, the answer is yes for bitolerance orders and no for tolerance orders.

2.) Proper and Unit bitolerance orders. Associated with any bitolerance order on X, there is a natural linear extension of that ordering: given

...s are closely related to tolerance graphs, ... see depth by Golumbic, Monma and Trotter. [ ] Given an assignment ... tolerance graph has an edge from x to y if and only if the overlap of ... define a bitolerance graph similarly; given an assignment of intervals and ... less than the center of I_y, we define x and y to ... as we ... of $t_\ell(x)$ and $t_r(y)$).

family of examples of proper tolerance graphs that are not unit tolerance graphs. In this section we briefly describe the order-theoretic approach to constructing those examples. For the sake of brevity, we omit details unless they are illuminating or differ significantly from [ ].

I guess I didn't get this written out in detail yet! That will be my project.

... incomparable with all of a, b

Ken Bogart's Ph.D. students

1. Issie Rabinovitch (1973) The Dimension theory of Semiorders and Interval Orders
2. T. Lockman Greenough (1976) Representation and Enumeration of Interval Orders and Semiorders
3. Jean Gordon (1977) Applications of the Exterior Algebra to Graphs and Linear Codes
4. Donald Goldberg (1978) Generalized Weights for Linear Codes
5. Anita Solow (1978) The Square Class Invariant for Quadradic Forms
6. Mark Halsey (1984) Line Closure and the Steinitz Exchange Axiom: Hartmanis Matroids
7. David Magagnosc (1987) Cuts and Chain Decompositions: Structure and Algorithms
8. Joseph Bonin (1989) Structural Properties of Dowling Geometries and Lattices
9. Karen Stellpflug Mandych (1990) Discrete representations of semiorders
10. Larry Langley (1993) Interval tolerance orders and dimension
11. Sanjay Rajpal (1995) On paving matroids representable over finite fields
12. Stephen Ryan (1999) Trapezoid orders: Geometric classification and dimension
13. Amy Myers (1999) Results in enumeration and topology of interval orders
14. Laura Montague Hegerle (2000) Join congruence relations on geometric lattices
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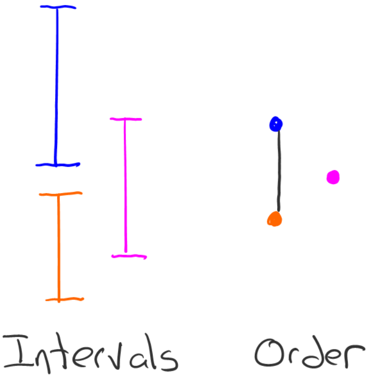

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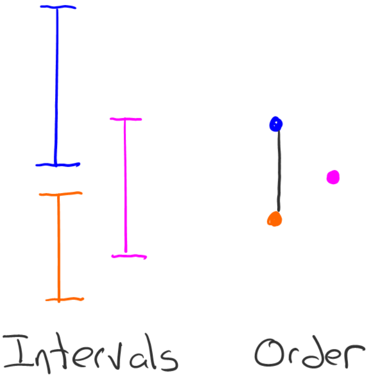

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- Intervals in other ordered sets
- Restrict Intervals
- Relax 'less than'

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- Equivalently - Tolerance Orders represented by ‘less than’ on a set of parallelograms with base on two lines
- There are many variations on tolerance orders/graphs
- Tolerance graphs are intersection graphs with tolerances. When there is a natural ordering they correspond to tolerance orders
- Bogart wrote a half dozen papers on tolerance graphs/orders

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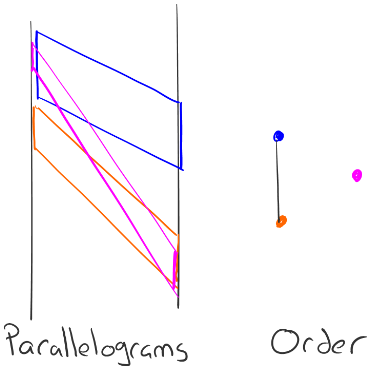
- Replace parallelogram with trapezoids to get interval dimension 2 orders (intersection of 2 interval orders)
- Bogart suggested extending bounded tolerance orders (parallelogram orders) to other geometric figures and 'more lines'. Results by Bogart and his students Ryan, Laison, Balof
- These problems can be viewed as investigating when a given order is a suborder of an (infinite) ordered set described by geometric objects.



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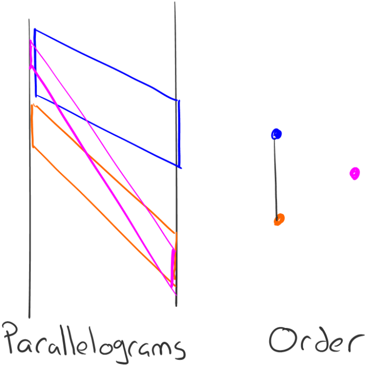

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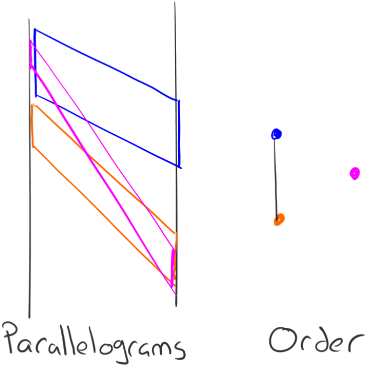

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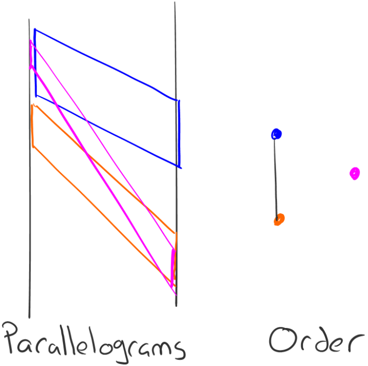

Restrictions on Intervals

Question posed by Bogart at a conference in 1989:

Given an interval order with specified bounds for each element is there an interval representation using intervals with integral endpoints and lengths within the specified bounds?

- Bogart's motivation: Interval orders can model intervals in time for scheduling, seriation etc
  This talk does not begin at 10:31.41597 and last 27.1828 minutes
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Related questions investigated by (at least)

- Fishburn (and Graham) 1983-1985: Bounded non-integral endpoints, minimize number of lengths, graphs
- Bogart and Stellpflug 1989: semiorders, minimum interval length
- Isaak 1990,1993: Bogart's question
- Pirlot 1990,1991: semiorders, minimum overall length
- Mitas 1994: semiorders, minimum interval length
- Pe'er and Shamir 1997: graph versions
- Myers 1999: interval orders, minimum overall length

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Create variables for left and right endpoints: $l(x)$, $r(x)$

- $x \prec y \Rightarrow r(x) + 1 \leq l(y)$

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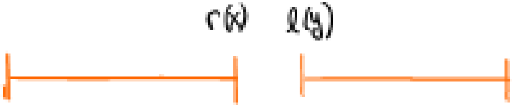

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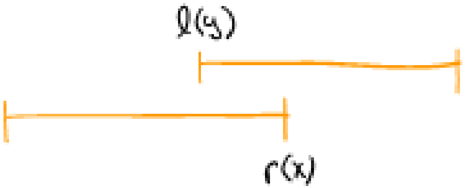

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  $\Rightarrow$ lower bound $\leq (r(x) - l(x)) \leq$ upper bound

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- This is a system of linear inequalities. The constraint matrix is totally unimodular so integrality does not need to be stated explicitly.
- Use linear programming algorithms to solve. Can add to minimize maximization to minimize overall length, total length of intervals etc.
- Use Farkas' Lemma/Theorem of the Alternative for necessary and sufficient conditions

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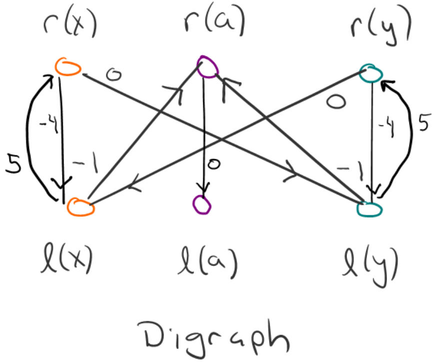
Bounded interval order representation ⇔ solution to a system of linear inequalities - via Bellman's equations solved by shortest paths in a related digraph



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- 'Characterizations' in terms of no negative cycles in the digraph translates to statements about the order
- Minimal forbidden suborders - harder in general
- With no upper bounds on length $\Rightarrow$ proof of interval order representation theorem (if and only if no $2+2$)
- Graph versions:
  - 'Equivalent' if lower bounds are all the same
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Minimal forbidden suborder lists:

- Non-discrete case (arbitrary endpoints)
  - represent with $1 \leq$ length $\leq k$ if and only if no $2+2$ and $1+(k+2)$
  - For rational q; represent with $1 \leq$ length $\leq q$ has a finite forbidden suborder list
  - For irrational r; represent with $1 \leq$ length $\leq r$ has a infinite forbidden suborder list
- Discrete case (integral endpoints) for semiorders, represent with intervals of length k. Can describe minimal forbidden suborders. Number of forbidden suborders is the Catalan number $\frac{1}{k+1}\binom{2k}{k}$.
  (non-discrete case covered by length 1 via scaling)
- Discrete case (integral endpoints)
  - represent with $0 \leq$ length $\leq k$ has a finite forbidden suborder list, minimal forbidden order list is ugly
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  - represent with $0 \leq \text{length} \leq k$ has a finite forbidden suborder list, minimal forbidden order list is ugly
  - represent with $1 \leq \text{length} \leq k$ has a infinite forbidden suborder list

Minimal forbidden suborder lists:

- Non-discrete case (arbitrary endpoints)
  - represent with $1 \leq$ length $\leq k$ if and only if no $2+2$ and $1+(k+2)$
  - For rational q; represent with $1 \leq$ length $\leq q$ has a finite forbidden suborder list
  - For irrational r; represent with $1 \leq$ length $\leq r$ has a infinite forbidden suborder list
- Discrete case (integral endpoints) for semiorders, represent with intervals of length k. Can describe minimal forbidden suborders. Number of forbidden suborders is the Catalan number $\frac{1}{k+1}\binom{2k}{k}$.
  (non-discrete case covered by length 1 via scaling)
- Discrete case (integral endpoints)
  - represent with $0 \leq$ length $\leq k$ has a finite forbidden suborder list, minimal forbidden order list is ugly
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Other Ground Sets

Question posed by Bogart:

Investigate orders based on intervals in other orders

- Investigated by Bogart, Bonin and Mitas (1994) and Mitas (1995)
- Bogart's motivation - 'Consider, for example the scheduling of meetings in rooms some distance apart with a discrete set of stopping and starting times, say every fifteen minutes. We can postulate that someone can travel from one room to another in one time period, but cannot participate in two meetings, one of which ends when the other starts , unless they are in the same room.'
- Another motivation - compact representations of ordered sets.
- Example above - ground set is weak orders. Question: what if in addition bounds are placed on interval lengths?

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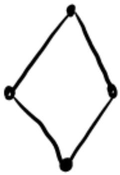
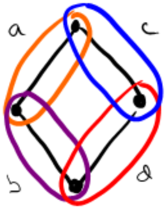
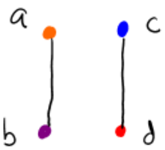
Order P is a Q based interval order:
Represent x and y by $[l(x), r(x)]$ and $[l(y), r(y)]$ and put $x \prec_P y$ if and only $r(x) \preceq_Q l(y)$



a weak order



intervals in a weak order



2+2 as intervals in weak order

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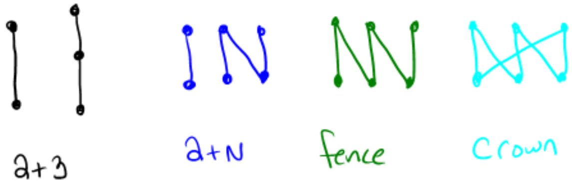
Bogart, Bonin and Mitas (1994) showed that P can be represented by intervals in a weak ($2+1$ free) order if and only if P does not contain a $3+2$, $N+2$, 6-element fence or a 6-element crown.



Other Ground Sets

Question posed by Bogart:

Investigate orders based on intervals in other orders

- Mitas (1995): forbidden suborder lists for interval orders in interval ($2+2$ free) orders (list of 37 forbidden suborders); and N free orders (list of 5 forbidden suborders).
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Conclusion

Bogart asked a number of interesting questions based on generic ideas of ‘intervals’ in ‘orders’ which resulted in work by a number of people