Degrees in Edge Colored Graphs

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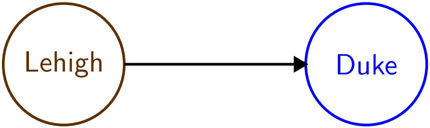
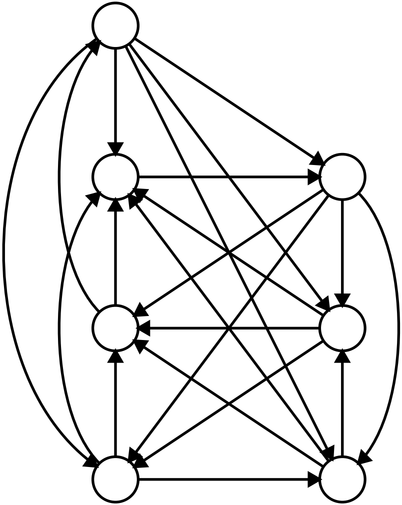
Will review a variety of results related to degrees in edge colored graphs

Not a survey - just some stuff interesting to me

Tournament



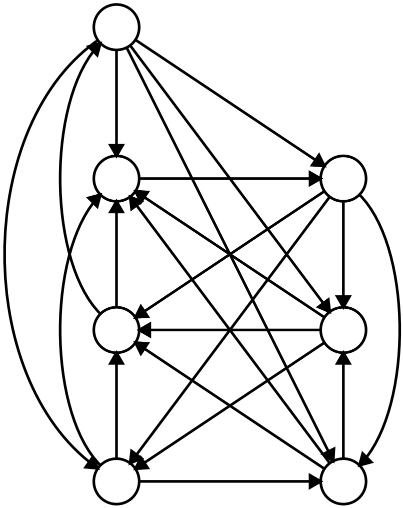
Interpret as Lehigh beats Duke



Score list

$(5, 4, 3, 3, 3, 2, 1)$

records number of wins



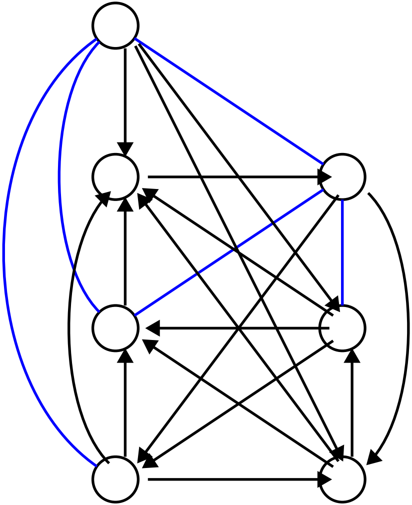
Score list
$(5, 4, 3, 3, 3, 2, 1)$
records number of wins

Necessary condition:

$$\sum_{i \in S} s_i \geq \binom{|S|}{2}$$

Wins for teams in S
as large as number of
games played among S

Landau (1954): Sufficient to be a score list



Allow Ties?

$$\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \cdots$$

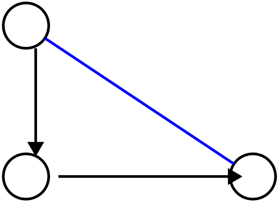
Win, Tie, Loss list

FIFA scores?

$12, 3, 9, \cdots$

$W = 3, T = 1, L = 0$

Necessary and sufficient conditions?



- Win, Tie, Loss Tournament = Oriented graph
- Oriented graph = digraph with no 2-cycles
- Characterization, algorithms for degree lists of these?

Degree lists of oriented graphs should be easy

- Easy to find an orientation of a given graph with specified degrees (network methods Ford and Fulkerson 1957)
- Degree lists of digraphs are characterized (= bipartitie degrees Gale, Ryser, Ore, 1957)
- Win minus loss records/ FIFA like totals with consecutive values are characterized (Avery 1991 tournaments, Mubayi, West, Will 2001 for digraphs/ from network methods)

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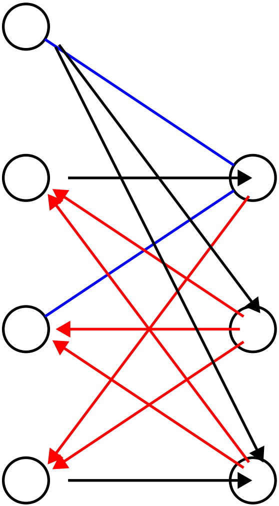
Degree lists of oriented graphs should be hard

- Bipartite oriented graph degree lists are NP-hard (Durr et al 2009 + Benz et al 2008)
- 'big wins', 'regular wins', ties records are NP-hard (same idea as bipartite oriented)
- Certain FIFA like completion problems are NP-hard (Pavlogi 2010)

Bipartite Tournament



= 3-edge colored
undirected bigraph

Is there a bipartite graph with given color vectors?

$(1, 1, 1, 1, 1)$ •

$(1, 1, 1, 1, 1)$ •

$(1, 1, 1, 1, 1)$ •

$(1, 1, 1, 1, 1)$ •

• $(2, 1, 1, 0, 0)$

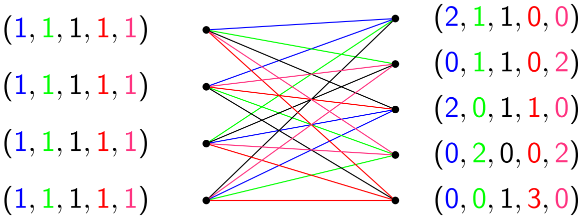
• $(0, 1, 1, 0, 2)$

• $(2, 0, 1, 1, 0)$

• $(0, 2, 0, 0, 2)$

• $(0, 0, 1, 3, 0)$

Is there a bipartite graph with given color vectors?



YES for this instance

In general its NP-hard

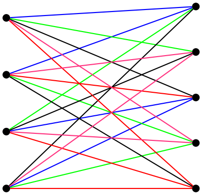
Row sums = column sums; use Birkhoff-VonNeumann Theorem

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$



$(2, 1, 1, 0, 0)$

$(0, 1, 1, 0, 2)$

$(2, 0, 1, 1, 0)$

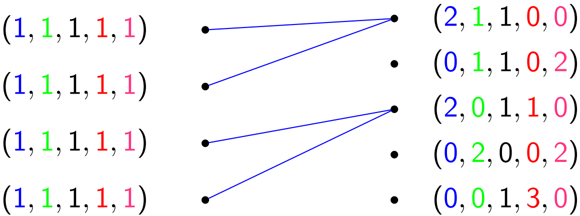
$(0, 2, 0, 0, 2)$

$(0, 0, 1, 3, 0)$

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This always works when one part degrees are 1's vectors



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‘Equivalent’ versions

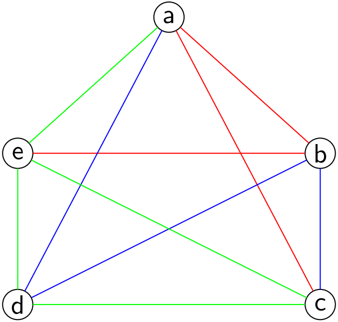
- Degree Lists of edge colored bipartite graphs
- Discrete tomography problem
- Restricted graph coloring/scheduling
- packing of graphic sequences
- special case of axial 3-way transportation

- Benz et al (2008) connection to discrete tomography - 5 color version is NP-hard from 2001 results
- Durr et. al. 2009 3-color version is NP-hard
- Results for a few special cases

Edge Colored Graphs

Look at (non-proper) edge colorings of complete graphs

	blue	green	red
a	1	1	2
b	2	0	2
c	1	2	1
d	2	2	0
e	0	3	1



Which sequences of vectors can be realized as degrees of an edge colored complete graph?

Columns - color sequences

Rows - color degrees

Edge Colored Graphs

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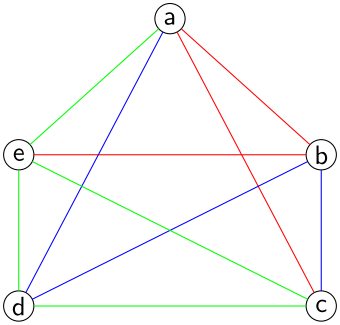


Necessary Condition - each color sequence is a degree sequence

≥ 4 colors: sums of color sequences must be degree sequences
Not sufficient

Necessary condition is sufficient (3 colors) when:

- One color sequence has all degrees $\in \{k, k+1\}$ (Kundu's Theorem, 1973)
  - extends to two outlying degrees and ...
- Two color sequences and their sum can be realized by forests (Kleitman, Koren and Li, 1977)
  - also a broader condition on the sum; characterize when both colors can be forests
- $\Delta \leq \sqrt{2}\delta n - \delta + 1$ where Δ, δ, min and max degree sum for two of the colors (Busch et al 2011)
- Two lists are identical (switch to get 'nice' Eulerian cycle in these colors then alternate)

8	6	2
8	0	8
7	4	5
5	2	9
4	4	8
3	1	12
.	.	.

8	0	8
8	1	7
7	2	7
3	4	9
4	4	8
3	6	7
.	.	.

Right is 'easier' than left

lower in Bruhat order is easier to realize

(Hartke and Seacrest 2011)

8	6	2
8	0	8
7	4	5
5	2	9
4	4	8
3	1	12
.	.	.

8	6	2
7	1	8
7	4	5
6	1	9
4	4	8
3	1	12
.	.	.

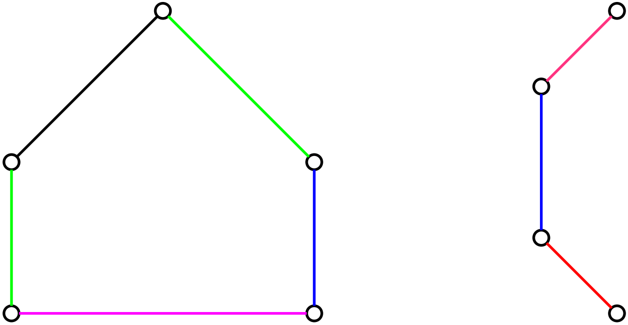
Right is 'easier' than left

'flatter is easier'

Necessary conditions are sufficient (multiple colors):

- All but 2 colors are $(1, 2, \ldots, 1)$ (i.e., 1-factors) Busch et al 2011
  - With at most 5 colors
  - or one of two remaining colors has all entries $\geq n/2$
- Sum of all but one color are forest realizable
  i.e., an edge colored forest with given degrees exists under obvious conditions
  - Inductive proof (Carroll 2009), requires pasting two smaller parts
  - Switching proof (Alpert, Becker, Hilbert, Iglesius REU 2010)
- One color has all degrees $\geq n - 4$
  i.e., an edge colored 3-regular graph with given degrees exists under obvious conditions
  infinite families of minimal 'forbidden graphs' for degree 4
  but for fixed number of colors finite number of examples
  (Alpert, Becker, Hilbert, Iglesius REU 2010)

- Idea of proof for forest realizations using switching
- complete proof uses another trick too
- Similar switching idea used for many of the results



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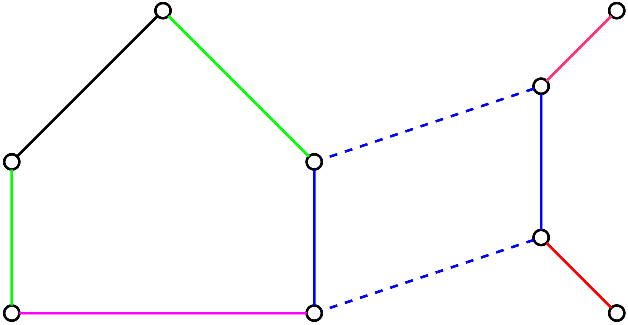
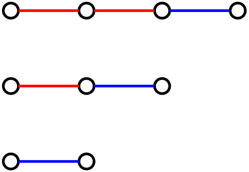


Alternate view

- Given list of k-tuples
- Given set of possible host graphs
- Can we k-color one of the host graphs to realize the list?
- Host sets
  - complete graph
  - complete bipartite graph
  - Forests
  - graphs with degree at most 3

What if host is a fixed linear forest
and we use just 2 colors?

Related problem: Given a list of degrees and a graph when does it have a subgraph with this degree list?



‘Degrees’: 2 - R; 4 - B; 2 - RB; 1 - RR; 0 - BB

Given path lengths $C_1, C_2, \ldots, C_p$ and numbers of each of 5 degree types when can we color to get these degrees? (Ryan - 2012+)

- If no *RB* vertices - NP-hard (depends on encoding) - exact subset sum
- Else assuming some counting conditions
  If R $\leq$ RB then
  if $RR \geq \sum_{i=1}^{(a-x)/2}(C_i - 2)$ and similar for BB
- Else assuming some counting conditions
  If R $\leq$ RB then previous conditions plus conditions on number of odd length paths