and hence that

$$\vdash . \text{inst}``\text{cs} \subset \text{ser}.$$

This shows us how we can construct a serial relation from any relation of the same sort as complete succession; or, indeed, from any relation agreeing with it in only one respect.

$*0 \cdot 1.\quad \vdash . \text{inst}``\hat{R}\{R \mid R_{se} \mid R \mathrel{\Subset} R\} \subset \text{ser}.$

Proof.

It is easy to show that

$$\vdash : \alpha \,\text{inst}`P\beta \,.\equiv .$$
$$\alpha = p`\overrightarrow{P}_{se}``\alpha \,.\, \beta = p`\overrightarrow{P}_{se}``\beta \,.\, (\exists x, y) \,.\, x \in \alpha \,.\, y \in \beta \,.\, xPy \qquad (1)$$

from the definitions of inst and τ_P. From this we can deduce

What's Your ORDER?

Wellesley College

October 6, 2015

Also, we find from (1) that

$$\vdash : \alpha \,\text{inst}`P\beta \,.\, \beta \,\text{inst}`P\gamma \,.\supset .$$
$$\alpha = p`\overrightarrow{P}_{se}``\alpha \,.\, \beta = p`\overrightarrow{P}_{se}``\beta \,.\, \gamma = p`\overrightarrow{P}_{se}``\gamma \,.$$
$$(\exists x, y, u, v) \,.\, x \in \alpha \,.\, y, u \in \beta \,.\, v \in \gamma \,.\, xPy \,.\, uPv.$$

This implies

$$\vdash : \alpha \,\text{inst}`P\beta \,.\, \beta \,\text{inst}`P\gamma \,.\supset .$$
$$\alpha = p`\overrightarrow{P}_{se}``\alpha \,.\, \gamma = p`\overrightarrow{P}_{se}``\gamma \,.\, (\exists x, v) \,.\, x \in \alpha \,.\, v \in \gamma \,.\, xP \mid P_{se} \mid Pv.$$

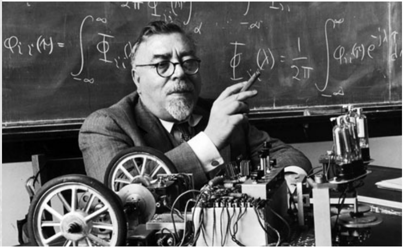
This, together with (1), gives us

$$\vdash . \text{inst}``\hat{R}\{R \mid R_{se} \mid R \subset R\} \subset \text{trans} \qquad (3)$$

By the definitions of inst and τ_P, we find that

- Interval Orders
  - Some History
  - Some Math
  - Why Intervals?
  - Computational Aside
- Linear Optimization
  - Some History
  - Some applications
  - Some Math
- Shortest Paths
  - Some Math
  - and a possibly a proof

Norbert Wiener (1894 - 1964)

- PhD from Harvard 1913 (age 18)
- Went to work with Bertrand Russell in Cambridge
- Wrote 3 papers on the theory of measurement in psychology
- Moved on to other areas of mathematics
- Papers lost, results reporved 1960's
- Papers rediscovered in 1990's by Fishburn

- Bertrand Russel and Alfred North Whitehead: Principia Mathematica (1910-1913)
- Attempt to develop all of mathematics from basic axioms
- Doomed in 1931 by Godel's Incompleteness Theorem

***110·632.** $\vdash : \mu \in \mathrm{NC} \,.\supset .\, \mu +_c 1 = \hat{\xi}\{(\exists y) \,.\, y \in \xi \,.\, \xi - \iota\text{‘}y \in \mathrm{sm}\text{“}\mu\}$

Dem.

$\vdash .\, *110{\cdot}631 \,.\, *51{\cdot}211{\cdot}22 \,.\supset$

$\vdash : \mathrm{Hp} \,.\supset .\, \mu +_c 1 = \hat{\xi}\{(\exists \gamma, y) \,.\, \gamma \in \mathrm{sm}\text{“}\mu \,.\, y \in \xi \,.\, \gamma = \xi - \iota\text{‘}y\}$

[*13·195] $= \hat{\xi}\{(\exists y) \,.\, y \in \xi \,.\, \xi - \iota\text{‘}y \in \mathrm{sm}\text{“}\mu\} : \supset \vdash .\, \mathrm{Prop}$

***110·64.** $\vdash .\, 0 +_c 0 = 0$ [*110·62]

***110·641.** $\vdash .\, 1 +_c 0 = 0 +_c 1 = 1$ [*110·51·61 . *101·2]

***110·642.** $\vdash .\, 2 +_c 0 = 0 +_c 2 = 2$ [*110·51·61 . *101·31]

***110·643.** $\vdash .\, 1 +_c 1 = 2$

Dem.

$\vdash .\, *110{\cdot}632 \,.\, *101{\cdot}21{\cdot}28 \,.\supset$

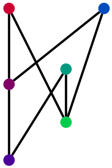
$\vdash .\, 1 +_c 1 = \hat{\xi}\{(\exists y) \,.\, y \in \xi \,.\, \xi - \iota\text{‘}y \in 1\}$

[*54·3] $= 2 \,.\supset \vdash .\, \mathrm{Prop}$

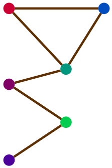
The above proposition is occasionally useful. It is used at least three times, in *113·66 and *120·123·472.

Intervals

$\Rightarrow$



Interval Order



Interval Graph

*A Contribution to the Theory of Relative Position**. By NORBERT WIENER, Ph.D. (Communicated by Mr G. H. Hardy.)

[*Received* 14 March 1914.]

The theory of relations is one of the most interesting departments of the new mathematical logic. The relations which have been most thoroughly studied are the *series*: that is, relations which are contained in diversity, transitive, and connected or, in Mr Russell's symbolism, those relations R of which the following proposition is true:

$$R \subset J \,.\, R^2 \subset R \,.\, R \cup \breve{R} \cup I \upharpoonright C'R = C'R \uparrow C'R.$$

Cantor, Dedekind, Frege, Schröder, Burali-Forti, Huntington, Whitehead, and Russell, are among those who have helped to give us an almost exhaustive account of the more fundamental properties of series. There is a class of relations closely allied to series, however, which has received very scant attention from the mathematical logicians. Examples of the sort of relation to which ... between two events in time when

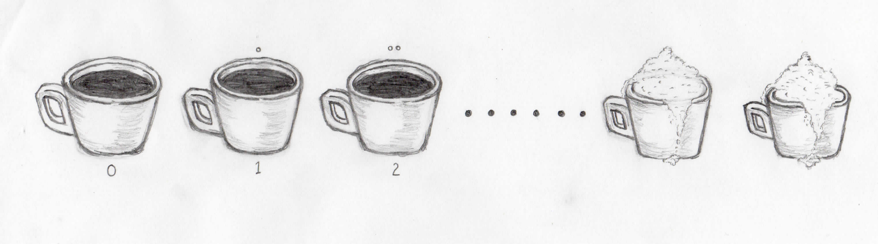
* The subject of this paper was suggested to me by Mr Bertrand Russell, and he paper itself is the result of an attempt to simplify and generalize certain notions ised by him in his treatment of the relation between the series of events and the eries of instants.

5. In conclusion, let us consider what bearing all this work of ours can have on experimental psychology. One of the great defects under which the latter science at present labours is its propensity to try to answer questions without first trying to find out just what they ask. The experimental investigation of Weber's law is a case in point: what most experimenters do take for granted before they begin their experiments is infinitely more important and interesting than any results to which their experiments lead. One of these unconscious assumptions is that sensations or sensation-intervals can be measured, and that this process of measurement can be carried out in one way only. As a result, each new experimenter would seem to have devoted his whole energies to the invention of a method of procedure logically irrelevant to everything that had gone before: one man asks his subject to state when two intervals between sensations of a given kind appear different; another bases his

whole work on an experiment where the observer's only problem is to divide a given colour-interval into two equal parts, and so on indefinitely, while even where the experiments are exactly alike, no two people choose quite the same method for working up their results. Now, if we make a large number of comparisons of sensation-intervals of a given sort with reference merely to whether one seems larger than another, the methods of measurement given in this paper indicate perfectly unambiguous ways of working up the results so as to obtain some quantitative law such as that of Weber, without introducing such bits of mathematical stupidity as treating a "just noticeable difference" as an "infinitesimal," and have the further merit of always indicating *some* tangible mathematical conclusion, no matter what the outcome of the comparisons may be.

Massachusetts Institute of Technology.

R. Duncan Luce Example (1956) from Econometrica



- Add one grain of sugar at a time
- Cannot distinguish between consecutive cups
- Can distinguish first and last
- Indifference is not transitive!

The traveling salesman meets the traveling archaeologist.

ALAN SHUCHAT
Wellesley College
Wellesley, MA 02181

An important part of an archaeologist's work is to assign dates to finds, or **deposits**. For example, the deposits may be graves in a prehistoric cemetery or strata in a trench dug at the site of an ancient settlement. The deposits may contain artifacts such as pottery, jewelry, or tools. Under very fortuitous circumstances, an inscription or other unambiguous characteristic of an artifact literally labels the deposit with a date. However, such luck is not common, and so archaeologists use mathematical and statistical techniques to help solve the problem of dating deposits. This article describes the fairly recent development of the use of matrices and networks to solve an ordering problem of particular interest to archaeologists called the **seriation problem**, and illustrates these techniques with two examples from the archaeological literature. These same ideas have been successfully applied in genetics, psychology, and management. Most of the work discussed is due to D. G. Kendall, E. M. Wilkinson, and G. Laporte, but the formulations and proofs supplied here are generally simpler than the original ones. An extensive survey of the problem from an archaeologist's point of view appears in [**22**], and an overview of the application of mathematics to archaeology can be found in [**6**], [**11**], and [**3**].

Reference to Petrie 1899

The seriation problem

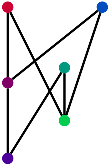
When archaeological deposits cannot be dated by deciphering inscriptions or applying physical or chemical techniques, but the deposits appear in stratifications, they may at least be placed in chronological order (a **seriation**). However, a site may have only partial stratification or none at all. In such cases, the archaeologist tries to order the deposits according to how their contents are related. For example, jewelry found in the deposits may show some progression of sophistication in manufacturing technique or style of decoration, and it is natural to associate this with a chronological ordering.

We will consider a collection of deposits containing artifacts that have been sorted into **types** by considering function, materials, shape, workmanship, style of decoration, etc. For example, a type may consist of all bronze anklets decorated with a certain arrow motif. We assume that

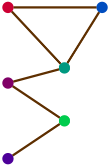
(**a**) each type corresponds to a time interval during which it was present, and

(**b**) each deposit corresponds to a point in time when it was formed (or at least to a time

Intervals

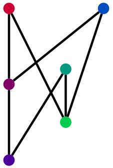


Interval Order



Interval Graph

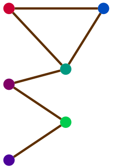
$\Rightarrow$



Intervals

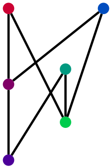
Interval Order

- Psychophysics
  - just noticeable difference
- Seriation in archeology
- Scheduling events
- DNA sequencing



Interval Graph

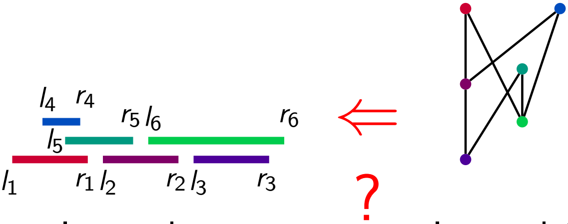
$\Leftarrow$?



Intervals

Interval Order

Given an order can we test if it is an interval Order?



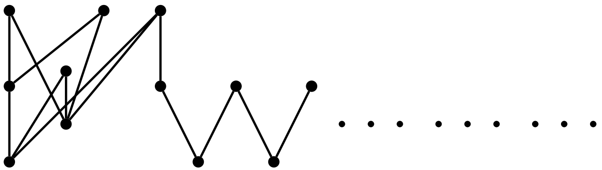
Intervals Interval Order

Algorithm for testing if an order has an interval representation: Check all possible orderings of endpoints

Check:

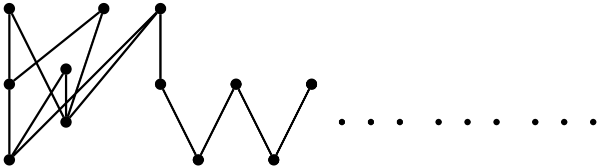
$$l_1\, l_4\, l_5\, r_4\, r_1\, l_2\, r_5\, l_6\, r_3\, l_3\, r_3\, r_6$$
$$l_1\, r_1\, l_4\, l_5\, r_4\, l_2\, r_5\, l_6\, r_3\, l_3\, r_3\, r_6$$
$$\vdots$$
$$l_4\, l_1\, l_4\, l_5\, r_4\, l_2\, r_5\, l_6\, r_3\, l_3\, r_3\, r_6$$

With 6 elements, check $\frac{12!}{2^6} = \frac{12 \cdot 11 \cdot 10 \cdots 2 \cdot 1}{2^6} = 7,484,400$ orderings



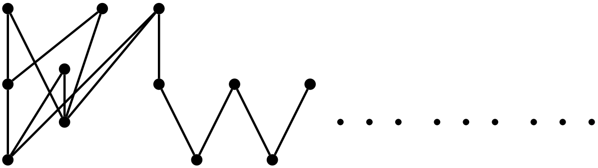
If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

- 1 second on my computer?
-
-
-



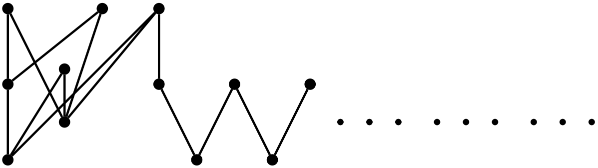
If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

- 1 second on my computer? NO
-
-
-



If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

- 1 second on my computer? NO
- 1 hour on my computer?
-
-



If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

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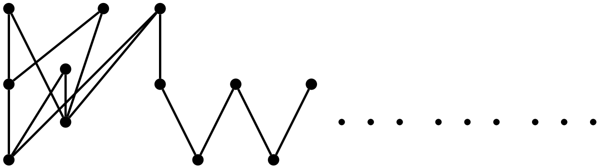
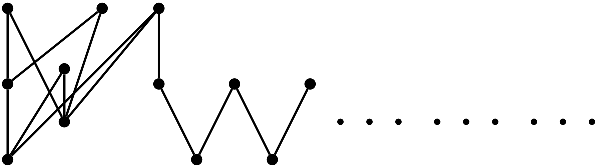
- 1 hour on my computer? NO
-
-



If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

- 1 second on my computer? NO
- 1 hour on my computer? NO
- 1 hour on all computers in this room?
-



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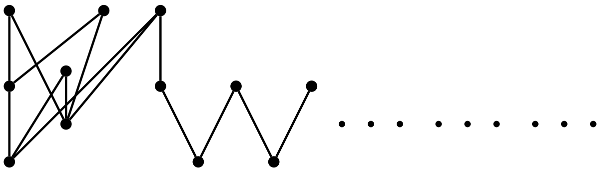
-



If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

- 1 second on my computer? NO
- 1 hour on my computer? NO
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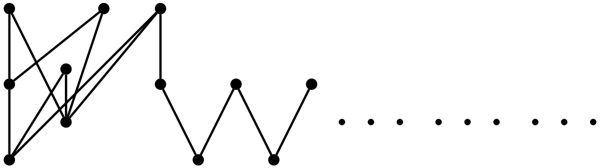
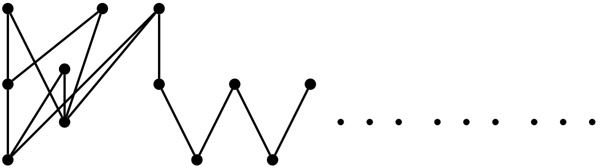
- 1 hour on all computers in the world?



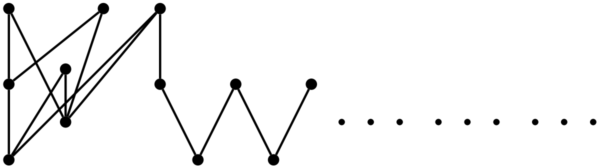
If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

- 1 second on my computer? NO
- 1 hour on my computer? NO
- 1 hour on all computers in this room? NO
- 1 hour on all computers in the world? NO



If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

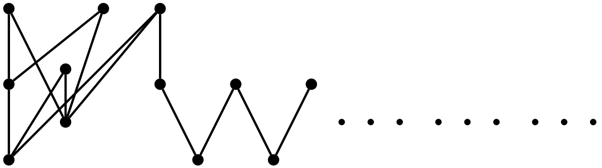
Can NOT test quickly
with all the computers in the world!



If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

Can NOT test quickly
with all the computers in the world!

True



If 37 elements: How long to test all $\frac{74!}{2^{37}}$ orderings?

> *Can NOT test quickly*
> *with all the computers in the world!*

True

Also true that Bill Gates is worth at least \$1, but ...

Check $\frac{74!}{2^{37}}$ orderings for interval representation

UNIVERSE-ALL computer:

All of the atoms in the known universe checking 100 billion orderings per second

Check $\frac{74!}{2^{37}}$ orderings for interval representation

UNIVERSE-ALL computer:

All of the atoms in the known universe checking 100 billion orderings per second

Still not done checking all possibilities for this instance

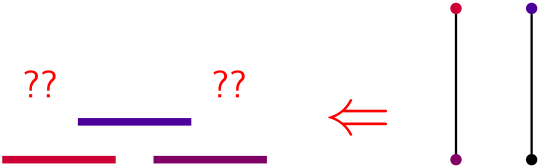
Check $\frac{74!}{2^{37}}$ orderings for interval representation

UNIVERSE-ALL computer:

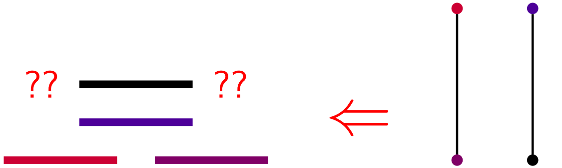
All of the atoms in the known universe checking 100 billion orderings per second

Still not done checking all possibilities for this instance

Use mathematical tools to make the check faster



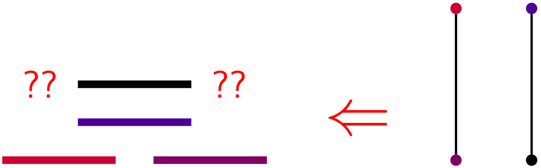
Intervals?? Interval Order



Intervals?? Interval Order

Lemma

Interval Order $\Rightarrow$ *NO* $2+2$



Intervals?? Interval Order

Lemma

Interval Order $\Rightarrow$ *NO* $2+2$

Theorem (Wiener 1914, Fishburn 1971)

Interval Order $\Leftrightarrow$ *NO* $2+2$

Theorem (Wiener 1914, Fishburn 1971)

Interval Order $\Leftrightarrow$ *NO* $2+2$

- Fast algorithm to test if there is an interval representation:
  Check all size 4 subsets to see if any is a $2+2$
- Millions of elements in a fraction of a second
- BUT: this doesn't provide a representation

and hence that

$$\vdash . \text{inst}``\text{cs} \subset \text{ser}.$$

This shows us how we can construct a serial relation from any relation of the same sort as complete succession; or, indeed, from any relation agreeing with it in only one respect.

$*0{\cdot}1.$ $\vdash . \text{inst}``\hat{R}\,\{R \mid R_{se} \mid R \Subset R\} \subset \text{ser}.$

Proof.

It is easy to show that

$\vdash : \alpha\, \text{inst}`P\beta . \equiv .$

$$\alpha = p`\overrightarrow{P}_{se}``\alpha . \beta = p`\overrightarrow{P}_{se}``\beta . (\exists x, y) . x \in \alpha . y \in \beta . xPy \qquad (1)$$

from the definitions of inst and τ_P. From this we can deduce

$$\vdash : \alpha\, \text{inst}`P\beta . \supset . \beta = p`\overrightarrow{P}_{se}``\beta . (\exists x, y) . x \in \alpha . y \in \beta . \sim xP_{se}y,$$

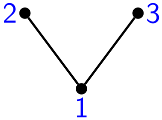
since by the definition of P_{se}, xPy and $xP_{se}y$ are incompatible. This reduces to

$$\vdash : \alpha\, \text{inst}`P\beta . \supset . \beta = p`\overrightarrow{P}_{se}``\beta . (\exists x) . x \in \alpha . \sim (x \in p`\overrightarrow{P}_{se}``\beta),$$

from which we can deduce

Model Interval Order using Inequalities

$$l_1 \le r_1 \qquad r_1 < l_2 \qquad \textit{not } r_2 < l_3$$
$$l_2 \le r_2 \qquad r_1 < l_3 \qquad \textit{not } r_3 < l_2$$
$$l_3 \le r_3$$



Rewrite as:

$$\begin{array}{rrrrrrl}
r_1 & & -l_2 & & & & \le -\epsilon \\
r_1 & & & & -l_3 & & \le -\epsilon \\
 & & & -r_2 & +l_3 & & \le 0 \\
 & & l_2 & & & -r_3 & \le 0 \\
l_1 & -r_1 & & & & & \le 0 \\
 & & l_2 & -r_2 & & & \le 0 \\
 & & & & l_3 & -r_3 & \le 0
\end{array}$$

We can represent an an order with intervals

$\Leftrightarrow$

Particular system of inequalities has a solution

Extends to:

- Constraints on interval length
- Minimize number of distinct endpoints
- Minimize ‘support’ length
  (if all lengths non-trivial)
- Partial information on ordering
- .....

WAIT

- We added ‘minimize’ to a system of inequalities
- Can we do that?
- Yes ......

$$\begin{array}{rrrrl} \max & 3x+4y+7z & & & \\ & x+y-2z & = & 1 \\ & -2x+y-3z & = & -3 \\ & x-5y+9z & = & 2 \end{array}$$

Uninteresting
Why??
...

$$\begin{array}{rrrrl} \max & 3x+4y+7z & & & \\ & x+y-z & \leq & 1 \\ & -2x+y-3z & \leq & -3 \\ & x-5y+9z & \leq & 2 \end{array}$$

Interesting
Nobel Prize
in Economics

Which has a solution?

$$\begin{aligned} x + y - 2z &\le 1 \\ -2x + y - 3z &\le -3 \\ x - 5y + 9z &\le 2 \end{aligned}$$

$$\begin{aligned} x + y - z &\le 1 \\ -2x + y - 3z &\le -3 \\ x - 5y + 9z &\le 2 \end{aligned}$$

Which has a solution?

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & \leq & 1 \\ -2x & + & y & - & 3z & \leq & -3 \\ x & - & 5y & + & 9z & \leq & 2 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & y & - & z & \leq & 1 \\ -2x & + & y & - & 3z & \leq & -3 \\ x & - & 5y & + & 9z & \leq & 2 \end{array}$$

Has a solution
for example $x = 1, y = 2, z = 1$

Which has a solution?

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & \leq & 1 \\ -2x & + & y & - & 3z & \leq & -3 \\ x & - & 5y & + & 9z & \leq & 2 \end{array}$$

Has a solution
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$$\begin{array}{rcrcrcr} x & + & y & - & z & \leq & 1 \\ -2x & + & y & - & 3z & \leq & -3 \\ x & - & 5y & + & 9z & \leq & 2 \end{array}$$

Has no solution

Which has a solution?

$$\begin{aligned} x + y - 2z &\le 1 \\ -2x + y - 3z &\le -3 \\ x - 5y + 9z &\le 2 \end{aligned}$$

Has a solution
for example $x = 1, y = 2, z = 1$

$$\begin{aligned} x + y - z &\le 1 \\ -2x + y - 3z &\le -3 \\ x - 5y + 9z &\le 2 \end{aligned}$$

Has no solution
Why not?

Show

$$\begin{array}{rcrcrcr} x & + & y & - & z & \leq & 1 \\ -2x & + & y & - & 3z & \leq & -3 \\ x & - & 5y & + & 9z & \leq & 2 \end{array}$$

Has no solution

Show

$$\begin{array}{rcrcrcr} x & + & y & - & z & \leq & 1 \\ -2x & + & y & - & 3z & \leq & -3 \\ x & - & 5y & + & 9z & \leq & 2 \end{array}$$

Has no solution

$$\begin{array}{l} 3\,(\;\; x + y - z \leq 1) \\ 2(-2x + y - 3z \leq -3) \\ 1\,(\;\; x - 5y + 9z \leq 2) \end{array}$$

Show

$$\begin{array}{rcrcrcr} x & + & y & - & z & \leq & 1 \\ -2x & + & y & - & 3z & \leq & -3 \\ x & - & 5y & + & 9z & \leq & 2 \end{array}$$

Has no solution

$$\begin{array}{l} 3\,(\quad x + y - z \leq 1) \\ 2(-2x + y - 3z \leq -3) \\ 1\,(\quad x - 5y + 9z \leq 2) \end{array} \Rightarrow$$

Show

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Has no solution

$$\begin{array}{l} 3\,(\quad x + \ y - \ z \leq \ 1) \\ 2(-2x + \ y - 3z \leq -3) \\ 1\,(\quad x - 5y + 9z \leq \ 2) \end{array} \Longrightarrow \begin{array}{rcrcrcr} 3x & + & 3y & - & 3z & \leq & 3 \\ -4x & + & 2y & - & 6z & \leq & -6 \\ x & - & 5y & + & 9z & \leq & 2 \\ \hline 0x & + & 0y & + & 0z & \leq & -1 \end{array}$$

Show

$$\begin{array}{rcrcrcr} x &+& y &-& z &\leq& 1 \\ -2x &+& y &-& 3z &\leq& -3 \\ x &-& 5y &+& 9z &\leq& 2 \end{array}$$

Has no solution

$$\begin{array}{l} 3\,(\quad x + y - z \leq 1) \\ 2\,(-2x + y - 3z \leq -3) \\ 1\,(\quad x - 5y + 9z \leq 2) \end{array} \Rightarrow \begin{array}{rcrcrcr} 3x &+& 3y &-& 3z &\leq& 3 \\ -4x &+& 2y &-& 6z &\leq& -6 \\ x &-& 5y &+& 9z &\leq& 2 \\ \hline 0x &+& 0y &+& 0z &\leq& -1 \end{array}$$

There is no solution

Else $0 \leq -1$

Show

$$\begin{array}{rcrcrcr} x & + & y & - & z & \leq & 1 \\ -2x & + & y & - & 3z & \leq & -3 \\ x & - & 5y & + & 9z & \leq & 2 \end{array}$$

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There is no solution

Else $0 \leq -1$

$u = 3$, $v = 2$, $w = 1$ is a certificate of inconsistency

Lemma (Farkas' Lemma 1906)

A system of inequalities has a solution
$\Leftrightarrow$ *it is not inconsistent*

$Ax \leq b$ *has a solution*
or $yA = 0,\ y \geq 0,\ yb < 0$ *has solution*

Can we extend Farkas' Lemma to Optimization?

$$
\begin{array}{rrrrrrrrr}
\max & 7x & - & 2y & + & z & & \\
 & x & + & y & - & 2z & \le & 1 \\
 & -2x & + & y & - & 3z & \le & -3 \\
 & x & - & 5y & + & 9z & \le & 2
\end{array}
$$

So max is at most 8

Aim is to find best multipliers

Called Shadow Prices/Lagrange Multipliers

Can we extend Farkas' Lemma to Optimization?

$$\begin{array}{rrrrrrrl} \max & 7x & - & 2y & + & z & & \\ & x & + & y & - & 2z & \leq & 1 \\ & -2x & + & y & - & 3z & \leq & -3 \\ & x & - & 5y & + & 9z & \leq & 2 \end{array}$$

$$\begin{array}{l} 7\,(\quad x + \ y - 2z \leq \ 1) \\ 1(\ -2x + \ y - 3z \leq -3) \\ 2\,(\quad x - 5y + 9z \leq \ 2) \end{array}$$

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\end{array}
$$

$$
\begin{array}{l}
7\,(\;\;x + y - 2z \leq 1) \\
1\,(-2x + y - 3z \leq -3) \\
2\,(\;\;x - 5y + 9z \leq 2)
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$$\begin{array}{l} 7\,(\quad x + y - 2z \leq 1) \\ 1\,(-2x + y - 3z \leq -3) \\ 2\,(\quad x - 5y + 9z \leq 2) \end{array} \Rightarrow \begin{array}{rrrrrrl} 7x & + & 7y & - & 14z & \leq & 7 \\ -2x & + & y & - & 3z & \leq & -3 \\ 2x & - & 10y & + & 18z & \leq & 4 \\ \hline 7x & - & 2y & + & z & \leq & 8 \end{array}$$

So max is at most 8

Aim is to find best multipliers

Called Shadow Prices/Lagrange Multipliers

$$
\begin{array}{lrcrcrcr}
\max & 7x & - & 2y & + & z & & \\
 & x & + & y & - & 2z & \leq & 1 \\
 & -2x & + & y & - & 3z & \leq & -3 \\
 & x & - & 5y & + & 9z & \leq & 2
\end{array}
$$

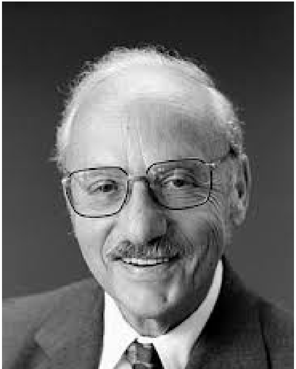
$$
\begin{array}{lrcrcrcr}
\min & u & - & 3v & + & 2w & & \\
 & u & - & 2v & + & w & = & 7 \\
 & u & + & v & - & 5w & = & -2 \\
 & -2u & - & 3v & + & 9w & = & 1 \\
 & u, & & v, & & w, & \geq & 0
\end{array}
$$

(Fundamental Theorem of Linear Programming Duality)

$$Min = Max$$

Linear Programming fast facts

- Name from ‘Program of resource Allocation’
  Before computer programs existed
- George Dantzig developed Simplex Method for solutions 1947
- 1975 Nobel Prize in Economics
  to Kantorovitch and Koopmans
  Koopmans threatened to boycott since Dantzig not included
- One of the most used algorithms today



Dantzig quote:

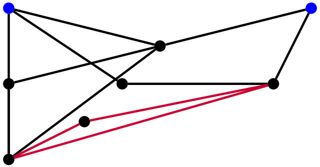
It happened because during my first year at Berkeley I arrived late one day at one of [Jerzy] Neyman's classes. On the blackboard there were two problems that I assumed had been assigned for homework. I copied them down. A few days later I apologized to Neyman for taking so long to do the homework — the problems seemed to be a little harder than usual. I asked him if he still wanted it. He told me to throw it on his desk. I did so reluctantly because his desk was covered with such a heap of papers that I feared my homework would be lost there forever. About six weeks later, one Sunday morning about eight o'clock, [my wife] Anne and I were awakened by someone banging on our front door. It was Neyman. He rushed in with papers in hand, all excited: "I've just written an introduction to one of your papers. Read it so I can send it out right away for publication." For a minute I had no idea what he was talking about. To make a long story short, the problems on the blackboard that I had solved thinking they were homework were in fact two famous unsolved problems in statistics. That was the first inkling I had that there was anything special about them.

Dantzig quote:

The other day, as I was taking an early morning walk, I was hailed by Don Knuth as he rode by on his bicycle. He is a colleague at Stanford. He stopped and said, "Hey, George — I was visiting in Indiana recently and heard a sermon about you in church. Do you know that you are an influence on Christians of middle America?" I looked at him, amazed. "After the sermon," he went on, "the minister came over and asked me if I knew a George Dantzig at Stanford, because that was the name of the person his sermon was about."

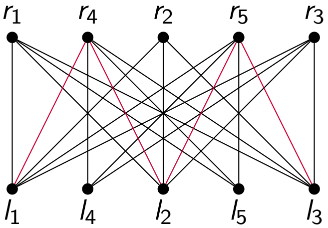
The origin of that minister's sermon can be traced to another Lutheran minister, the Reverend Schuler [sic] of the Crystal Cathedral in Los Angeles. He told me his ideas about thinking positively, and I told him my story about the homework problems and my thesis. A few months later I received a letter from him asking permission to include my story in a book he was writing on the power of positive thinking. Schuler's published version was a bit garbled and exaggerated but essentially correct. The moral of his sermon was this: If I had known that the problem were not homework but were in fact two famous unsolved problems in statistics, I probably would not have thought positively, would have become discouraged, and would never have solved them.

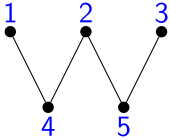
Shortest Paths



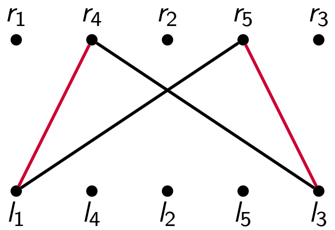
- Find shortest path to each vertex
- Can also be modeled with linear inequalities
- Makes sense iff there are no negative ‘cycles’
- Iff corresponds to multipliers for inequalities

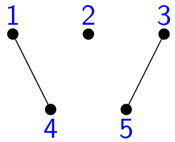
Shortest Paths and Interval Orders





Shortest Paths and Interval Orders





(Weiner 1914, Fishburn 1971)

Interval Order $\Leftrightarrow$ *NO* $2+2$

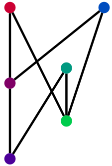
'Proof':

- Formulate as solving a system of linear inequalities
- Observe that the system corresponds to a shortest path problem
- Show that negative cycles $\Rightarrow$ there is a $2+2$

Proof has advantage of extending to bounds on interval lengths, partial information, minimizing support....

Shortest Paths and Interval Orders

⇐



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Intervals