Distinguishing number of line graphs of complete bipartite graphs

Or: Distinguishing number > 2

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Permute rows and columns to get original array?

1	2	1	2
2	1	1	2
2	1	2	1

1	1	1	2
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Permute rows and columns to get original array?

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row 1 ⇕ row 3

2	1	2	1
2	1	1	2
1	2	1	2

column 2 ⇔ column 3
column 1 ⇔ column 4

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row ‘degrees’ distinct
columns distinct
$\Rightarrow$ only trivial permutations

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Bottom array is an identity coloring - only automorphism is the trivial one

Permute rows and columns to get original array?

$$\begin{array}{cccc} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{array} \quad \begin{array}{c} \text{row 1} \\ \Updownarrow \\ \text{row 3} \end{array} \quad \begin{array}{cccc} 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{array} \quad \begin{array}{c} \text{column 2} \Leftrightarrow \text{column 3} \\ \text{column 1} \Leftrightarrow \text{column 4} \end{array} \quad \begin{array}{cccc} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{array} \quad \begin{array}{c} \text{row `degrees' distinct} \\ \text{columns distinct} \end{array} \Rightarrow \text{only trivial permutations}$$

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Determine the smallest c such that an $s \times t$ grid has an identity coloring

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Determine the smallest c such that an $s \times t$ grid has an identity coloring - fewest colors to destroy symmetry

Determine the smallest t such that every c coloring of $s \times t$ grid has a non trivial automorphism - smallest 'size' with unavoidable symmetry

Determine the smallest c such that an $s \times t$ grid has an identity coloring

Determine the distinguishing number of the cartesian product of two complete graphs: $D(K_s \Box K_t) =$? Fewest colors on vertices such that only trivial automorphism

Determine fewest colors on edges of complete bipartite graph such that only trivial automorphism

Determine the distinguishing number of the action of the group $S_s \times S_t$ on the grid $\mathbb{N}_s \times \mathbb{N}_t$

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Imrich, Jerebic, Klavzar (2008) formula for most values, recursion for a few

Determine fewest colors on edges of complete bipartite graph such that only trivial automorphism

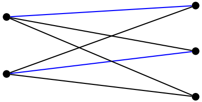
Fisher and Isaak (2008) formula for most values, recursion for a few

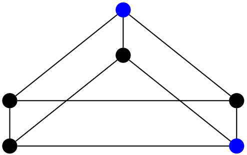
Determine the distinguishing number of the action of the group $S_s \times S_t$ on the grid $\mathbb{N}_s \times \mathbb{N}_t$

Chan (2006) recursion for values

Permute rows and columns to get original array?

$$\begin{matrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{matrix}$$





Identity coloring of $s \times t$ grid

$\Leftrightarrow$ identity edge coloring of $K_{s,t}$ (bipartite adjacency matrix)

$\Leftrightarrow$ identity vertex coloring of $K_s \Box K_t$ (Imrich and Miller result on automorphisms of $G \Box H$)

Identity 2-coloring of 2×2 grid: $\begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix}$

Identity 2-coloring of 3×3 grid: $\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{matrix}$

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- Need 3 colors if we allow transposes

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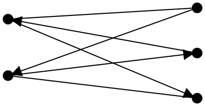
- Have automorphisms if we allow transposes
- Need 3 colors if we allow transposes
- Otherwise $s = t$ cases we can 'ignore' transposes

Identity orientations

Orient some of the edges of G $\rightarrow$ only trivial automorphism

'Destroying symmetry by orienting edges: complete graphs and complete bigraphs' (Harary and Jacobson 2001)

also: Harary and Robinson (2001) and Harary and Ranjan (2005)
For bipartite graphs



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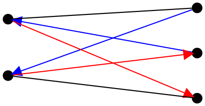
For bipartite graphs $=$ 3 colors



Cartesian products and distinguishing number

$G \square H$ - replace vertices of G with copies of H, edges of G become edges between copies of the same vertex of H

G is prime if $G = G_1 \square G_2$ only trivially

$G \square G \square G = G^3$ etc

- Hypercubes - $D(K_2^n) = 2$ except $D(K_2^2) = D(K_2^3) = 3$ (Bogstead and Cowen 2004)
- Connected prime graphs - $D(G^r) = 2$ for $r \geq 4$ (Albertson 2005)
- Connected graphs - $D(G^r) = 2$ for $r \geq 3$ (Klavzar and Zhu 2007)
- $D(G^r) = 2$ for $r \geq 2$ except $D(K_2^2) = D(K_2^3) = D(K_3^2) = 3$ (Imrich and Klavzar 2006)

G, H connected relatively prime graphs with $|G| \neq |H|$

$\Rightarrow D(G \square H) \leq D(K_{|G|} \square K_{|H|})$

automorphisms of $G \square H$ are also automorphisms of $K_{|G|} \square K_{|H|}$

(Imrich, Jerebic and Klavzar 2008)

Bounds on distinguishing number of Cartesian products of complete graphs are general bounds

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With 2 colors how big can t be?

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- Columns must be distinct else permute duplicates

1	2	1	1	2	
1	1	1	2	1	permute columns 2 and 5
1	2	2	2	2	

$\Rightarrow t \leq 2^3$ (else duplicated column)

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  1 1 1 2 1 permute columns 2 and 5
  1 2 2 2 2

  $\Rightarrow t \leq 2^3$ (else duplicated column)

- $t = 2^3$

  1 1 1 1 2 2 2 2
  1 1 2 2 1 1 2 2
  1 2 1 2 1 2 1 2

  Any row permutation $\Rightarrow$ grid with all possible columns
  $\Rightarrow$ some column permutation gives original grid
  $\Rightarrow t \leq 2^3 - 1$

Find an identity coloring of a $3 \times t$ grid ($t \geq 3$)

With 2 colors how big can t be?

- $t = 2^3 - 1$?

1	1	1	1	2	2	2	2
1	1	2	2	1	1	2	2
1	2	1	2	1	2	1	2

- Omit any column - apply row permutation that preserves the column

  $\Rightarrow$ grid with all columns except omitted column

  $\Rightarrow$ some column permutation gives original grid

  $\Rightarrow t \leq 2^3 - 2$

Find an identity coloring of a $3 \times t$ grid ($t \geq 3$)
With 2 colors how big can t be?

- $t = 2^3 - 2$?

1 1 2 2 2 2
1 2 1 1 2 2
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Find an identity coloring of a $3 \times t$ grid ($t \geq 3$)
With 2 colors how big can t be?

- $t = 2^3 - 2$?

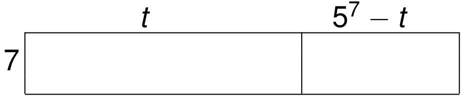
  1 1 2 2 2 2
  1 2 1 1 2 2
  1 1 1 2 1 2

- row 'degrees' distinct, columns distinct $\Rightarrow$ only trivial permutations

  $\Rightarrow t = 2^3 - 2$ has an identity 2-coloring

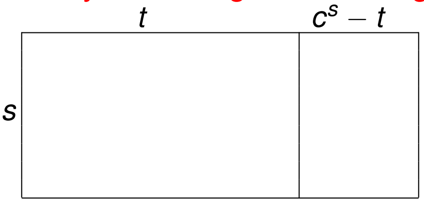
Switching lemma example

5 colors and 7 rows: Complete to all 5^7 possible columns



- Non-trivial automorphism on right $\Leftrightarrow$ Non-trivial automorphism on right
- Switching Lemma: $s \times t$ grid has an identity c coloring if and only if $s \times c^s - t$ grid does.

Identity c coloring of an $s \times t$ grid ($t \geq s$)?



- If t is 'small' use 'distinct row degrees', distinct columns construction
- If t is 'big' (close to c^s) use induction and switching lemma and look for identity coloring of $s \times c^s - t$ grid

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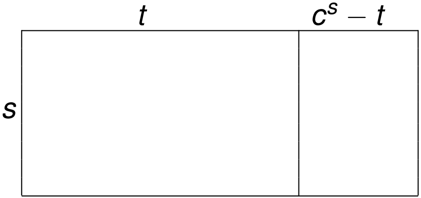

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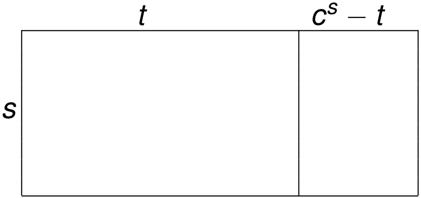

- YES if $t \leq c^s - \lceil \log_c s \rceil - 1$
- NO if $t \geq c^s - \lceil \log_c s \rceil + 1$
- SOMETIMES if $t = c^s - \lceil \log_c s \rceil$

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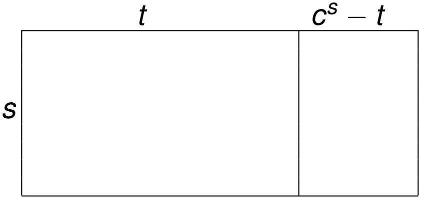

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- SOMETIMES if $t = c^s - \lceil \log_c s \rceil$
  - NO if $s \geq c^{1+x} - \lfloor \log_c x \rfloor$
  - YES if $s \leq c^{1+x} - \lfloor \log_c x \rfloor - 2$
  - SOMETIMES if $s = c^{1+x} - \lfloor \log_c x \rfloor - 2$
  - $x = \lfloor \log_c (s-1) \rfloor$

Translation to distinguishing number for $s \leq t$:

Let $c = \lceil (t+1)^{1/s} \rceil$
smallest c such that $t < c^s$

$D(K_s \Box K_t) =$

$$\begin{cases} c+1 \text{ if } c^s - \lceil \log_c s \rceil + 1 \leq t \leq c^s - 1 \\ c \text{ if } s \leq t \leq c^s - \lceil \log_c s \rceil \\ c \textit{ or } c+1 \text{ when } t = c^s - \lceil \log_c s \rceil \text{ (determine by recursion)} \end{cases}$$

It will be c if t is 'big' relative to s; e.g. $t \geq s^s$

Grids in higher dimensions?
Example: Note that $13 \cdot 42 = 546$

- $13 \times 42 \times (c^{546} - \lceil \log_c 546 \rceil - 1)$ grid has an identity c coloring

  (identity c coloring of $546 \times (c^{546} - \lceil \log_c 546 \rceil - 1)$ grid)

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- $13 \times 42 \times c^{546} - \lceil \frac{\log_c 42}{13} \rceil - 1$ grid has an identity c coloring (3-dimension version of switching lemma)

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General case replace $13, 42$ with s, t

Summary - we can 'most of the time' directly determine the distinguishing number of $K_s \square k_t$ and otherwise can determine with a recursion using at most iterated logarithm (base c of t) steps. Extends to higher dimensional grids