Hamiltonian Path Variants in Structured Graph Families

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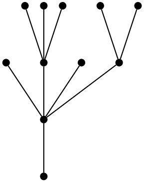
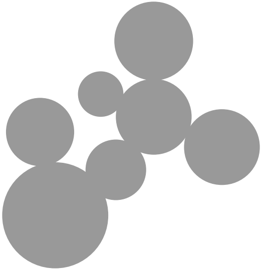
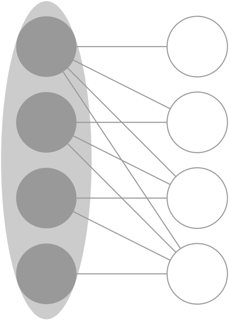
Highly Structured Families



Tree



Block Graph



Threshold Graph

General strategy:

- *NP hard problem*
- *Polynomial on structured families*
  For our families sometimes trivially polynomial
- *Find certifying algorithm/ structure theorem*

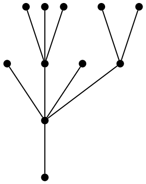
Explanation of result notation:

- Fact - well known = old, probably wrong if attempt attribution
- Fact - exercise = possibly new, undergraduate homework level
- Proposition = possibly new and requires some work

Note - no attempt to survey

Hamiltonian Path = spanning path

Does T have a Hamiltonian Path?



Hamiltonian Path = spanning path

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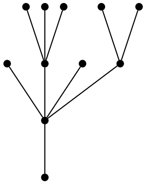

Well NO

Hamiltonian Path = spanning path

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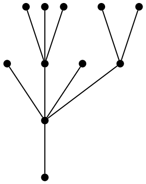
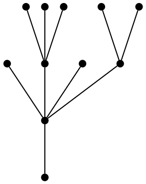


Well NO WHY?

Hamiltonian Path = spanning path

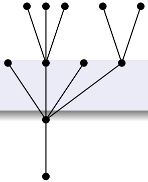
Does T have a Hamiltonian Path?



Well NO   WHY?

- *Each leaf must be an end*
- *cut vertex splits graph into >2 components*

How close to Hamiltonian is T?



Hamiltonian Path = Spanning Path

- *Spanning PathS - minimum number*
- *Spanning Walk - minimum ?*

Minimum Number of Paths needed to span vertices?

Disjoint (vertex and edge) = Path Partition - $PP(G)$
often called path cover

Edge Disjoint Cover (vertices may repeat)

Path Cover (vertices and edges may repeat) = $PC(G)$

For today skip Edge (but not vertex) disjoint

Different values for different path cover versions

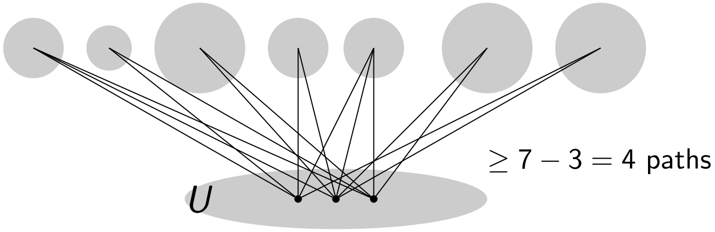
Minimum Spanning Walk?

$Minimize\ length = MLSW(G)$

$Minimize\ maximum\ vertex\ repeat = MRSW(G)$

All min repeat spanning walks are ‘long’
and
All min length spanning walks, vertex with ‘many’ repeats

Reminder of basic bound for path partition - Scattering Number



Fact (Well known)

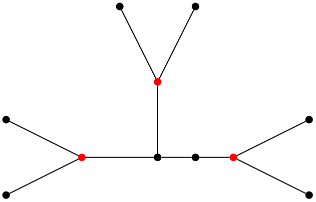
Min Path Partition $\geq$ *Max* $C(G - U) - |U|$

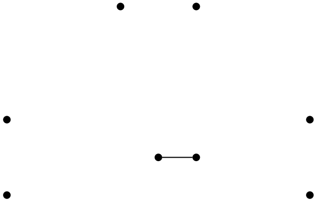
Equality for trees, threshold graphs, co-comparability graphs,...

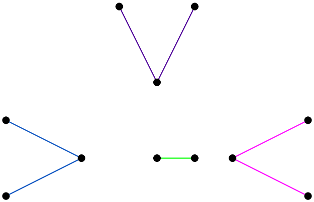
Goal - get nice minimax thms for Hamiltonian variants on structured classes

TREES

Well Known: Min Path Partition = Max $C(G - U) - |U|$



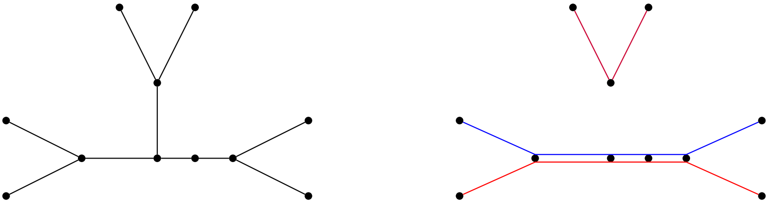




TREES

Exercise: Min Path Cover $= \left\lceil \frac{\#\text{leaves}}{2} \right\rceil$

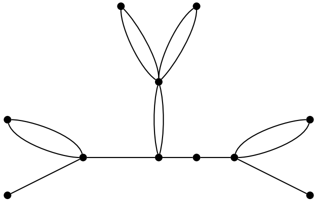
Hararay and Schwenk (1972): can cover edges with same number



TREES

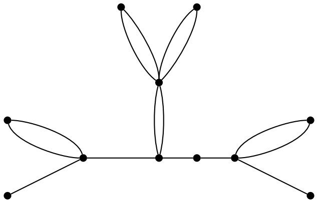
Well Known: Tree on n vertices

Min length spanning walk $= 2(n-1) - diameter(T)$



TREES

Fact: Tree maximum degree Δ has
Min repeats in spanning walk $= \Delta$ or $\Delta - 1$



$MRSW(T) = \Delta - 1$ if some path contains all degree Δ vertices
$MRSW(T) = \Delta$ otherwise

TREES

Fact

If T is a tree

- $PP(T) = \max C(G-U) - |U|$ *(scattering number)*
- $PC(T) = \left\lceil \frac{\#leaves}{2} \right\rceil$
- $MLSW(T) = 2(n-1) - diameter(T)$
- $MRSW(T) = \Delta - 1$ *or* Δ ...

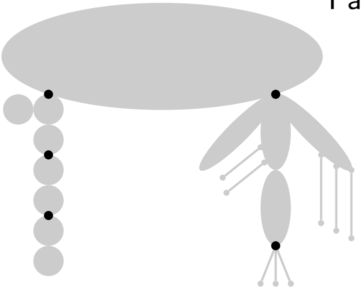
BLOCK GRAPHS

Fact

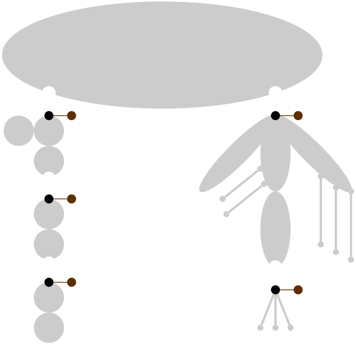
If G is a block graph with blocks B_i and Δ_B the maximum number of blocks containing a cut vertex

- $PP(G)$ *is* NOT *the scattering number. .....*
- $PC(T) = \left\lceil \frac{\#endblocks}{2} \right\rceil$
- $MLSW(T) = \sum |B_i| - diameter(G)$
- $MRSW(T) = \Delta_B - 1$ *or* Δ_B ...

Path Partition in Block Graphs

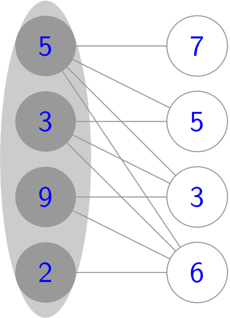


Decompose on certain cut vertices
count end blocks and ...



THRESHOLD GRAPHS

Well known: Min Path Partition = Max $C(G-U)-|U|$



$7+1-5=3$

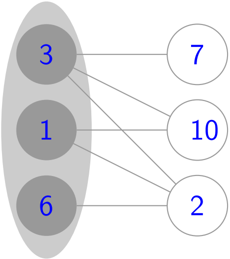
$12+1-8=5$

$15+1-17=-1$

$21-19=2$

THRESHOLD GRAPHS

Fact: $LB \leq$ Min Path Cover $\leq UB$



LB

UB

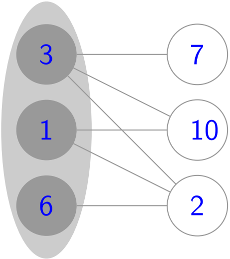
$\frac{7}{3} = 3$

$\frac{17}{4} = 5$

$\frac{19}{10} = 2$

THRESHOLD GRAPHS

Fact: $LB \leq$ Min Path Cover $\leq UB$

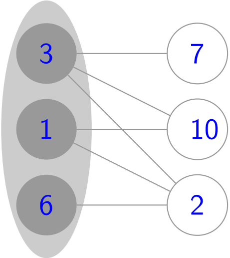


LB	UB
$\frac{8}{4} = 2$	$\frac{7}{3} = 3$
$\frac{18}{5} = 4$	$\frac{17}{4} = 5$
$\frac{19}{11} = 2$	$\frac{19}{10} = 2$

- $4 \leq PC(G) \leq 5$
- Bottom greedy algorithm yields optimal cover

THRESHOLD GRAPHS

Fact: $LB \leq$ Min Path Cover $\leq UB$



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$\frac{8}{4} = 2$	$\frac{7}{3} = 3$
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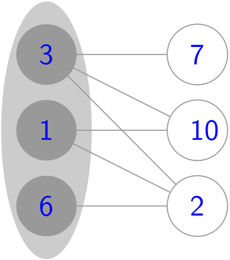
- LB, UB gap can be arbitrarily large
- If $LB \geq Gap^2$ then full spectrum of values

Threshold graph path cover algorithm:

THRESHOLD GRAPHS

Fact: Min repeats in spanning walk = $\max \left\lceil \frac{C(G-S)-1}{|S|} \right\rceil$



LB

$\frac{7}{3} = 3$

$\frac{17}{4} = 5$

$\frac{18}{10} = 2$

$MRSW(G) = 5$

THRESHOLD GRAPHS

Fact: Min length spanning walk $= n - 2 + PP(G)$

True in general for diameter 2 graphs

THRESHOLD GRAPHS

Fact

If G is a threshold graph

- $PP(T) = \max C(G-U) - |U|$ *(scattering number)*
- $LB \leq PC(T) \leq UB$ ....
- $MLSW(T) = n - 2 + PP(G)$
- $MRSW(T) = \max \left\lceil \frac{C(G-U)-1}{|U|} \right\rceil$