Voting Profiles, Constant Row Tomography, Edge Coloring Regular Bipartite Multigraphs,3-Dimensional Contingency Table Reconstruction, Birkhoff-Von Neumann Theorem, ...

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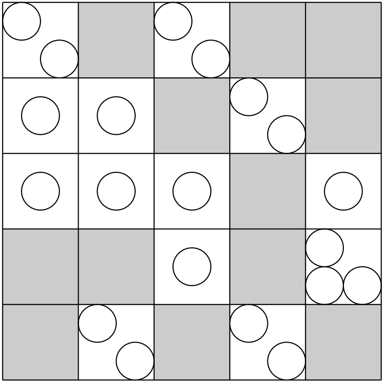
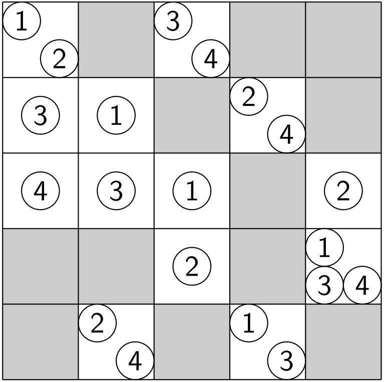


Fill ○'s with $\{1, 2, 3, 4\}$

Distinct entries per Row/Column



Can be done with any choice of 5 and 4

Even 4 > 5

Konig (1916), Frobenius (1917),
Birkhoff (1946), Von Neumann (1953), ...

today stick with 5 and 4 for all examples

A	B	C	D	E	
1, 2		3, 4			Abel
3	1		2, 4		Bernoulli
4	3	1		2	Cayley
		2		1, 3, 4	Descartes
	2, 4		1, 3		Euler

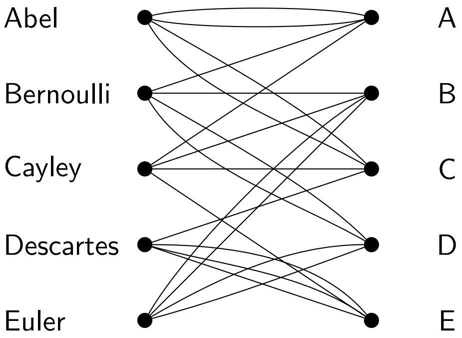
5 Professors, 5 classes, 4 days

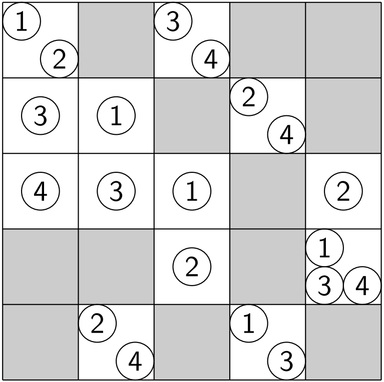
1 class per professor per day

Choices of classes that add up correctly

Translate to graph version:
record number of ○ for bipartite adjacency matrix

2	0	2	0	0
1	1	0	2	0
1	1	1	0	1
0	0	1	0	3
0	2	0	2	0



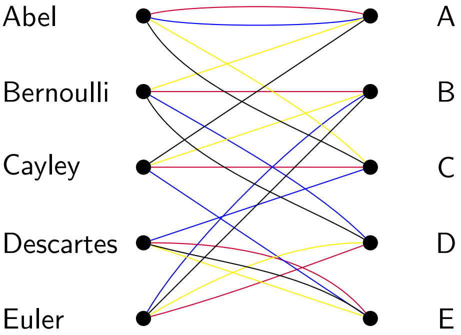


$$\begin{matrix} 2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 2 & 0 \end{matrix} = \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} + \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix} + \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} + \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$$

Konig's Theorem:

Use permutation matrices for proper edge coloring

$$\begin{matrix} 2&0&2&0&0\\ 1&1&0&2&0\\ 1&1&1&0&1\\ 0&0&1&0&3\\ 0&2&0&2&0 \end{matrix} = \begin{matrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{matrix} + \begin{matrix} 1&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 0&1&0&0&0 \end{matrix} + \begin{matrix} 0&0&1&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{matrix} + \begin{matrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0 \end{matrix}$$



Birkhoff - Von Neumann Theorem :
Every Doubly Stochastic matrix is
a convex combination of permutations matrices.

$$\begin{matrix} 2&0&2&0&0\\ 1&1&0&2&0\\ 1&1&1&0&1\\ 0&0&1&0&3\\ 0&2&0&2&0 \end{matrix} = \begin{matrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{matrix} + \begin{matrix} 1&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 0&1&0&0&0 \end{matrix} + \begin{matrix} 0&0&1&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{matrix} + \begin{matrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0 \end{matrix}$$

For BVN, Scale so row/column sums are 1.

Konig Theorem scaled is rational version of BVN.

Real might be 'harder' than rational but not in this case

4 Voters rank 5 candidates:
Abel, Bernoulli, Cayley, Descartes, Euler:

V1	V2	V3	V4
A	A	C	C
B	D	A	D
C	E	B	A
E	C	E	E
D	B	D	B

$\Rightarrow$ Profile:

	A	B	C	D	E
1st	2	0	2	0	0
2nd	1	1	0	2	0
3rd	1	1	1	0	1
4th	0	0	1	0	3
5th	0	2	0	2	0

Reconstruct a voting profile:

	A	B	C	D	E
1st	2	0	2	0	0
2nd	1	1	0	2	0
3rd	1	1	1	0	1
4th	0	0	1	0	3
5th	0	2	0	2	0

$\Leftarrow$

V1	V2	V3	V4
A	A	C	C
B	D	A	D
C	E	B	A
E	C	E	E
D	B	D	B

Use permutation matrices to determine votes:

$$\begin{matrix} 2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 2 & 0 \end{matrix} = \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} + \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix} + \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} + \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$$

Discrete tomography:
'measure' color counts in each row/column and try to reconstruct

A	A	C	C	$(2,0,2,0,0)$
B	D	A	D	$(1,1,0,2,0)$
C	E	B	A	$(1,1,1,0,1)$
E	C	E	E	$(0,0,1,0,3)$
D	B	D	B	$(0,2,0,2,0)$
$(1,1,1,1,1)$	$(1,1,1,1,1)$	$(1,1,1,1,1)$	$(1,1,1,1,1)$	

NP-hard in general

$=$ Voting profiles if column counts are $(1,1,1,\ldots,1)$, i.e. columns are permutations

A	A	C	C	(2,0,2,0,0)
B	D	A	D	(1,1,0,2,0)
C	E	B	A	(1,1,1,0,1)
E	C	E	E	(0,0,1,0,3)
D	B	D	B	(0,2,0,2,0)
(1,1,1,1,1)	(1,1,1,1,1)	(1,1,1,1,1)	(1,1,1,1,1)	

0-1 indicator variables $X_{i,j,k}$: 1 if i, k entry is color j

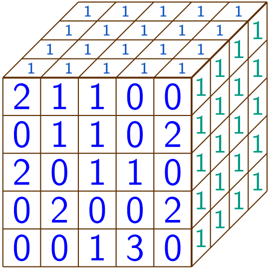
$\sum_k X_{i,j,k} = 1$: Each column is permutation

$\sum_j X_{i,j,k} = 1$: Each cell gets 1 color

$\sum_i X_{i,j,k} =$ row color count

All variants have similar $X_{i,j,k}$ encodings

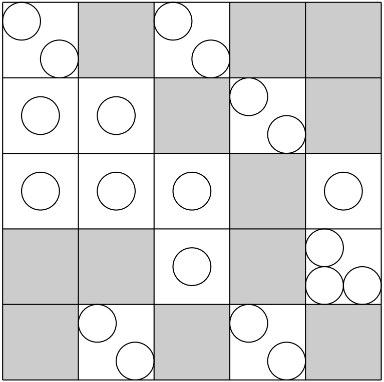
= 3-dimensional contingency table reconstruction



NP-hard in general

Always a solution if 2 ‘sides’ have marginals all $= 1$

‘Turn’, i.e., swap subscripts to get new variants



'List' version has size 4 lists (possibly different) for each ○

Still require distinct entries per Row/Column

'harder' to prove that always works

Uses stable matching

All variants have similar $X_{i,j,k}$ encodings