A Two Player Graph Pebbling Game

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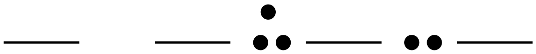
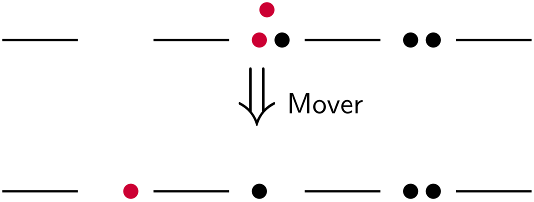
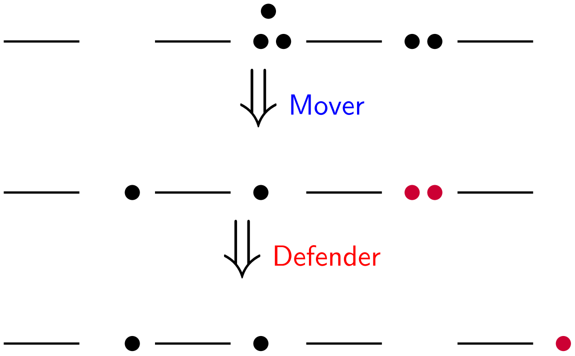


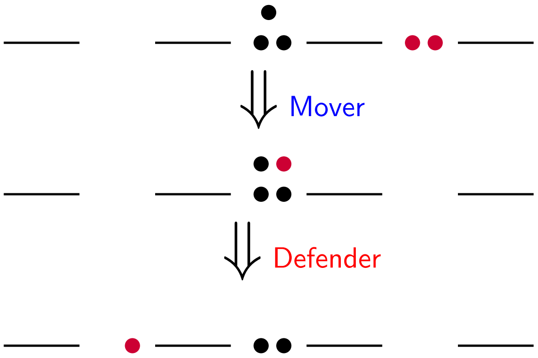
- Two players: Mover and Defender
- Pick 2 pebbles, discard 1, place 1 in adjacent pit
- Mover wins if pebble placed in left pit
- Defender wins if no moves remain
  and Mover has not won


Mover

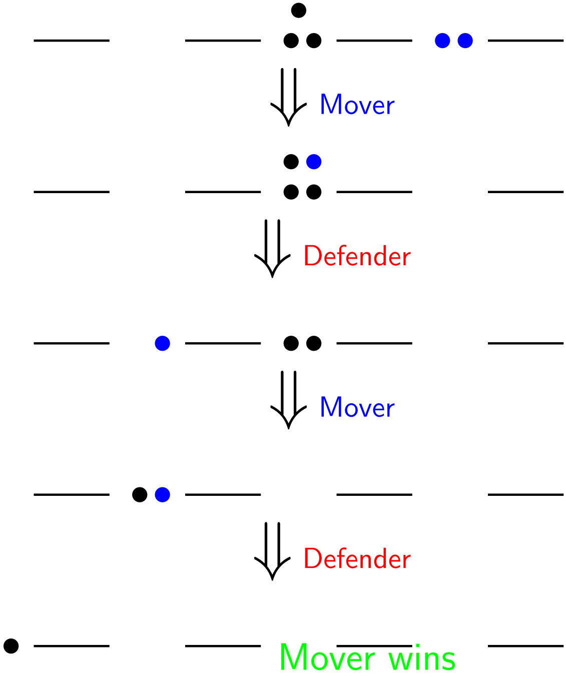


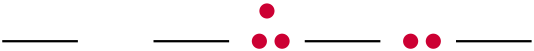
Defender wins if mover plays from middle first

Different strategy for Mover



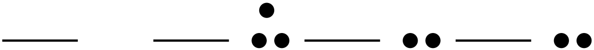
- Defender cannot undo last move
- Neither player can pass


Mover
Defender
Mover
Defender
Mover wins



Mover has a winning strategy

- Two players: Mover and Defender
- Pick 2 pebbles, discard 1, place 1 in adjacent pit
- Defender cannot undo last move
- Neither player can pass
- Mover wins if pebble placed in left pit
- Defender wins if no more moves

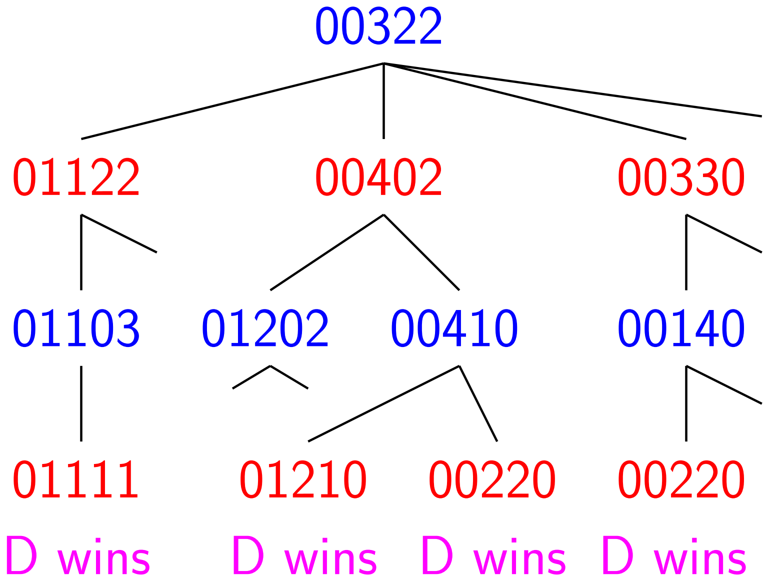


$$=$$

$$0 \quad 0 \quad 3 \quad 2 \quad 2$$

Mover *wins* 0 0 3 2 0

?? *wins* 0 0 3 2 2


00322
01122
00402
00330
01103
01202
00410
00140
01111
01210
00220
00220
D wins
D wins
D wins
D wins

Mover wins 0 0 3 2 0

Defender wins 0 0 3 2 2

- Given an initial configuration which player has a winning strategy?
- Find η:
  Mover WINS ALL configurations with $\geq \eta$ pebbbles
  Defender WINS SOME configuration with $\eta - 1$ pebbles
- Even if all pebbles on far end?

Classical Pebbling: only one player

$$0 \quad 0 \quad 3 \quad 2 \quad 0$$

- Potential: $P = \frac{0}{1} + \frac{0}{2} + \frac{3}{4} + \frac{2}{8} + \frac{0}{16} = 1$

Fact

On paths WIN if and only if $P \geq 1$

Proof: Invariant under pebbling toward root

Fact

Classical pebbling on paths: $\pi = 2^{n-1}$

WIN for ALL configurations with $\geq 2^{n-1}$ pebbles
LOSE for SOME configuration on $2^{n-1} - 1$ pebbles

Game Pebbling

? Fact ?

Mover wins on paths if potential $P \geq 2$

FAKE Proof:
Two players, double the potential...

? Fact ?

$\eta(P_n) = 2 \cdot 2^{n-1} = 2^n$

FAKE Proof:
Mover uses classical pebbling strategy with 2^{n-1} pebbles.
while Defender wastes the other 2^{n-1}

Theorem

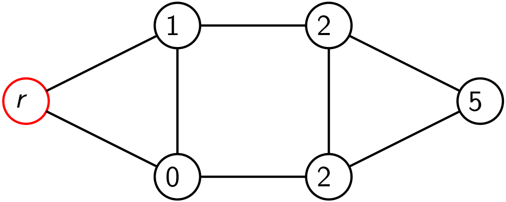
$$\eta(P_n) \leq \frac{3}{2} \cdot 2^{n-1} - n$$

'large' number of pebbles mover can win in at most $\frac{3}{2} \cdot 2^{n-1}$ plays

but if number of pebbles close to η then defender is forced to help mover

induction and .....

What is exact value of $\eta(P_n)$?

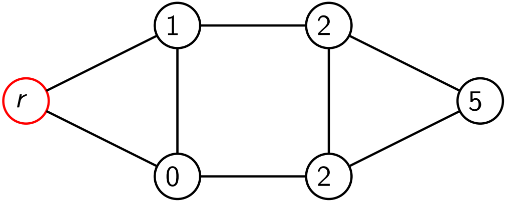


- Which player has winning strategy for given configuration?
- Min # pebbles so Mover always wins (for given root)
- Min # pebbles so Mover always wins (for all roots)

Min might not exist

e.g. Make center C_4 a K_4 and min does not exist

Classical Pebbling



- Only one 'player'
- Min # pebbles to get to root (always finite)
- Introduced for number theory problem
  $g_1, g_2, \ldots, g_n$ elts in $\mathbf{Z}_n$ order $|g_i|$
  there is nonempty zero sum subset with $\sum \frac{1}{|g_i|} \leq 1$
- Many variants

Two player Graph pebbling game

$$\eta(G, v)$$

- Mover wins* ALL configurations $\geq \eta$ pebbles
- Defender wins* SOME $\eta - 1$ pebble configuration

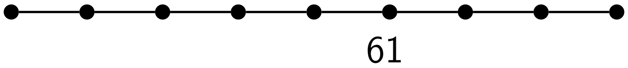
$$\eta(G)$$

- $\eta(G)$ maximum over all roots of $\eta(G, v)$

η can be infinite

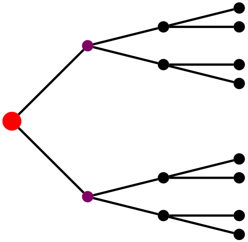
* wins = has a winning strategy

2 Player graph pebbling on 'infinite' paths



- Min # pebbles for Mover to win if all pebbles on first n vertices?
- Between 2^n and $\frac{3}{2} \cdot 2^n - 2n$ (probably)
  Recall $\eta(P_n) \leq \frac{3}{2} \cdot 2^{n-1} - n$
- Will help with $\eta(P_n)$
- Will help with cycles, is $\eta(C_n) \sim 2^{n/2}$?

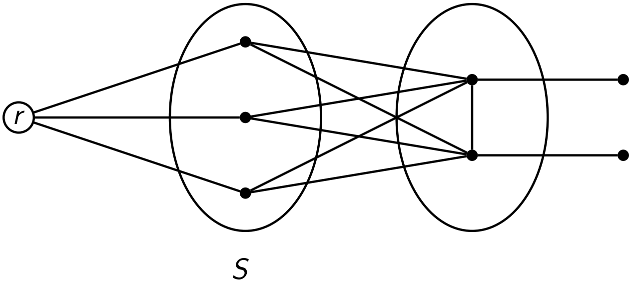
Trees



$\eta(T) = \infty$ *if .... e.g. some v and 2 of its neighbors all have degree at least 3*

Which trees have finite $\eta(T)$?

Sufficient condition for $\eta = \infty$



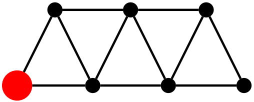
Cut set S all vertices degree ≥ 2 'out'

All neighbors of S degree ≥ 2 'out'

'most' graphs have $\eta = \infty$ but ....

Paths: $2^{n-1} < \eta(P_n) \leq \frac{3}{2} \cdot 2^{n-1} - n$

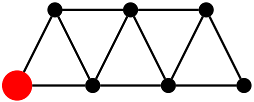
Squares of Paths?



P_7^2

Paths: $2^{n-1} < \eta(P_n) \leq \frac{3}{2} \cdot 2^{n-1} - n$

Squares of Paths?



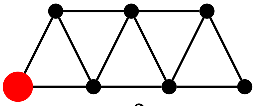
P_7^2

Square of a path: $\eta(P_n^2, v_1) = \infty$ *if* $n \geq 7$

Does not use cutset lemma, need to do more for proof

Paths: $2^{n-1} < \eta(P_n) \leq \frac{3}{2} \cdot 2^{n-1} - n$

Squares of Paths?



P_7^2

Square of a path: $\eta(P_n^2, v_1) = \infty$ *if* $n \geq 7$

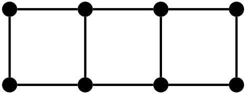
$\eta(P_n^k, v_1) = \infty$ *if* $2 \leq k \leq n - 5$

Path powers have $\eta = \infty$ except possibly those close to complete graphs

Path powers 'close' to complete graphs

- $\eta(K_n) = n$ (trivially)
- $\eta(P_n^{n-2}, v_1) = 2n - 2$ i.e., $\eta(K_n - e) = 2n - 2$
- $\eta(P_n^{n-3}, v_1) = 3n - 7$, i.e., $\eta(K_n - P_3) = 3n - 7$
- $\eta(P_n^{n-4}, v_1)$ is finite $(4n - c$ for some $c)$
  K_n with C_4 with 2 pendant edges removed
- Path powers $\eta(P_n^k, v_1) = \infty$ if $2 \leq k \leq n - 5$

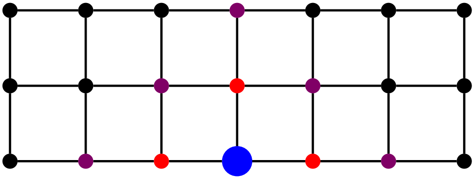
Grids?



$P_4 \square P_2$

Ladders: $\eta(P_n \square P_2)$ *finite, exact value?*

Grids?



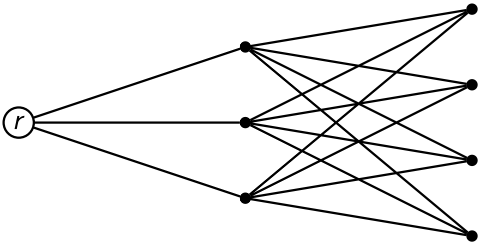
$P_7 \square P_3$

Grids $\eta(P_m \square P_n) = \infty$ *for* $m \geq 3, n \geq 5$ *from cutset lemma*

$\eta(P_n \square P_3, v_1)$ *finite (probably), exact value?*

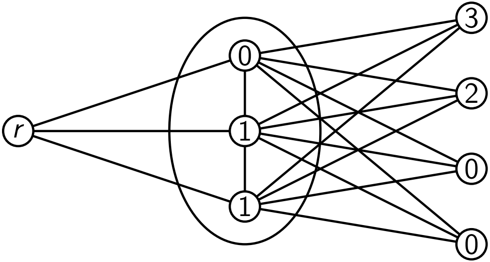
Cutset lemma $\Rightarrow \eta = \infty$
Look at graph classes where cutset lemma does not apply

Complete bipartite graphs

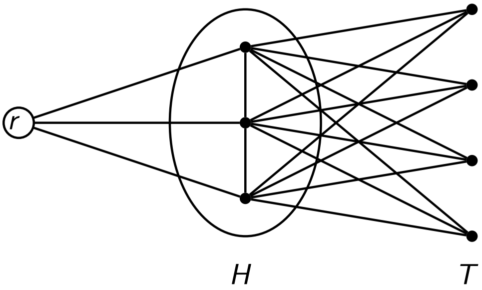


- vertices in cutset have degree ≥ 2 'out'
- BUT neighbors of cutset have all edges 'in' to cutset
- Defender is forced to play to neighbors of root
- Hope for winning strategy for Mover

Certain Diameter 2 graphs

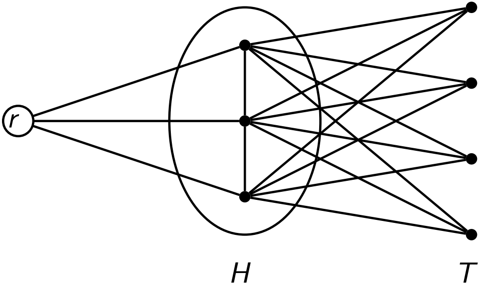


Certain Diameter 2 graphs



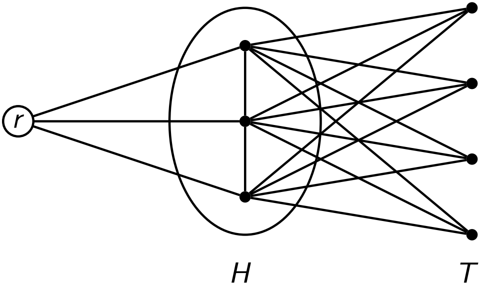
- H arbitrary $\qquad k$ pebble free vertices in H
- $C_T = \sum \lfloor \frac{c(v)}{2} \rfloor$ v in T
- k odd: Mover 'wins' if $C_T \geq k+1$
- k odd: Defender 'wins' if $C_T \leq k$

Certain Diameter 2 graphs



- H arbitrary $\quad$ k pebble free vertices in H
- $C_T = \sum \lfloor \frac{c(v)}{2} \rfloor$ v in T
- k even: Mover 'wins' if $C_T \geq k+3$
- k even: Defender 'wins' if $C_T \leq k+1$

Certain Diameter 2 graphs



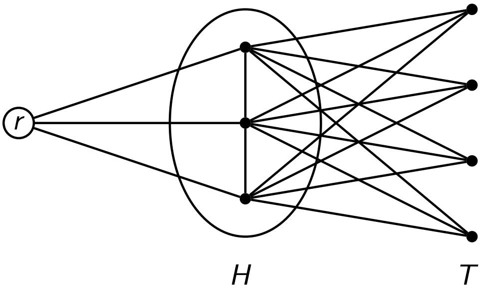
- H arbitrary; k pebble free vertices in H; k even
  $C_T = \sum \lfloor \frac{c(v)}{2} \rfloor = k + 2$
- Mover wins if exactly one 'odd'vertex in T
- Defender wins if at least 2 even or 1 zero vertex in T

If: k pebble free vertices in H; k even $C_T = \sum \lfloor \frac{c(v)}{2} \rfloor = k + 2$
and
exactly one vertex even (and nonzero) number of pebbles in T
Then depends on structure of H and 'equivalent' to ...

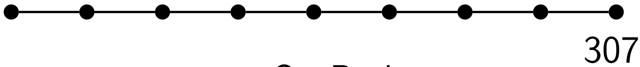
Element Selecting Game

- Subsets $N_1, N_2, \ldots, N_k$ of U
- Mary and Dan take turns picking an element from U and deleting it from all N_i
- Mary wins when some set becomes empty
- How many turns for Mary to win?

Certain Diameter 2 graphs



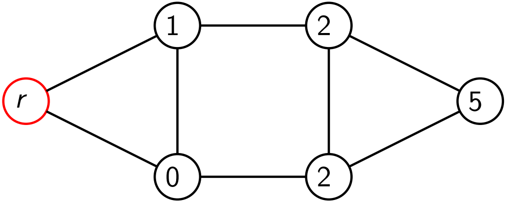
- Need element selecting game to determine winner for certain configurations
- But we can get $\eta(G,r) = |T| + 2|H| = 4$ ($|H|$ even)
  $\eta(G,r) = |T| + 2|H| = 3$ ($|H|$ odd)
- Complete multipartite: $\eta(G) = 2n - a + 3$
  if smallest part size is $a \geq 3$ and $n - a$ even



On Paths

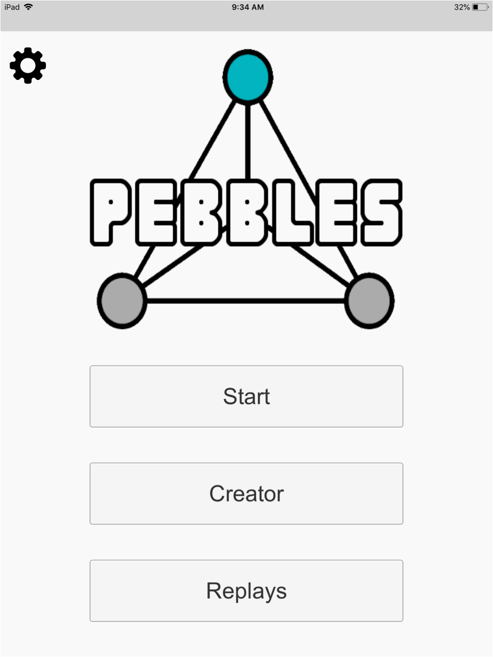
- $\eta \leq \frac{3}{2} 2^{n-1} - n$ but what is exact value?
- Paths: Who wins if all pebbles on last vertex?
  All pebbles on some intermediate vertex?
- Can we describe optimal strategy for either player?
- Paths: Is potential $P \geq 2$ sufficient for Mover to win?
  or some other sufficient or necessary conditions?
- Complexity of determining winner for configuration?

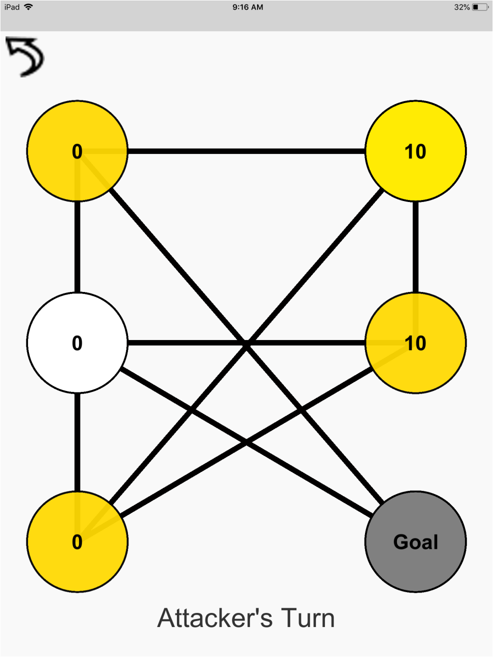
2 Player Graph Pebbling Game



- $\eta = \infty$ for 'many' graphs, formally show 'almost all' graphs ...
- η finite for paths, cycles, ladders, P_n^{n-4}:
  determine exact values, complexity, point configurations ...
- Other nice families? Trees, ... ?
- Say more about element selecting game
  If each elt missing from exactly one set?
  needed for multipartite with small part

Play Pebbles on Apple




iPad
9:16 AM
32%
0
10
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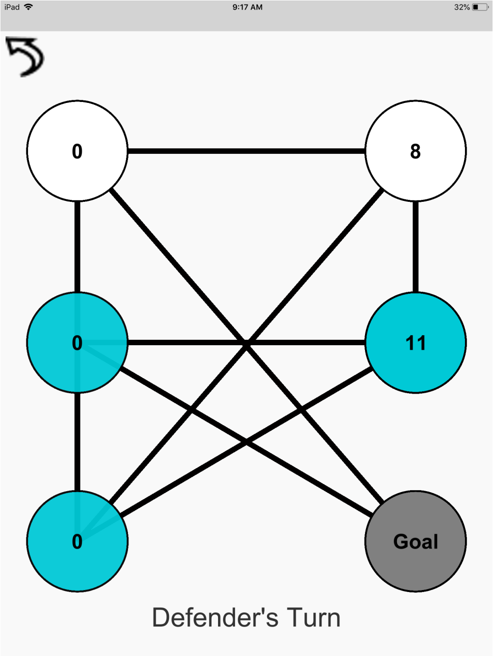
10
0
Goal
Attacker's Turn


iPad
9:17 AM
32%
0
8
0
11
0
Goal
Defender's Turn