Equivalence Class Universal Cycles for Permutations

Glenn Hurlbert* Garth Isaak†

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1 Introduction

In this note we describe a representation of permuations of an n-element set that can be viewed as equivalence classes of permuations of length n on $n+1$ symbols. An *equivalence class universal cycle* is a string $x_1x_2,\ldots,x_{n!}$ such that among the $n!$ length n substrings $x_ix_{i+1},\ldots x_{i+n}$ (subscript addition modulo $n!$) each equivalence class is represented exactly once. We produce a *complete family* of n such cycles. In such a family, distinct cycles use distinct representatives and each member of an equivalence class acts as representative exactly once.

The notion of universal cycles as cyclic representations of combinatorial objects, as a generalization of DeBruijn cycles, was introduced by Chung, Diaconis and Graham [1] and studied by Hurlbert [2] and Jackson [3]. The universal cycles for permutations that we examine here are one such example.

Let $\Pi^k_{i,j}$ denote the set of all $k-$permutations of $\{i,i+1,\ldots,j\}$. We write a typical element $\mathbf{a}\in\Pi^k_{i,j}$ as a vector of k distinct terms from $\{i,i+1,\ldots,j\}$. It is easy to show that universal cycles exist for $\Pi^k_{1,n}$ for $1\leq k\leq n-1$ and do *not* exist for $k=n$. (See Jackson [3].) Chung, Diconnis and Graham [1] use the concept of order isomorphism as an equivalence relation on strings from $\Pi^n_{1,m}$ to get universal cycles for $\Pi^n_{1,n}$. Such cycles exist for $m\geq 5/2n$ (Hurlbert [3]) and it is conjectured that $m=n+1$ suffices. We consider another natural equivalence relation on $\Pi^n_{0,n}$ for which equivalence class universal cycles representing $\Pi^n_{1,n}$ exist. This gives a universal cycle for permuations of length $\{1,2,\ldots,n\}$ using $n+1$ symbols. Moreover, we are able to construct a complete family of such cycles.

*Depatment of Mathematics, Arizona State University, Tempe, AZ
†Department of Mathematics, Lehigh University, Bethlehem, PA

2 Equivalence Classes

Let $\mathbf{1}_m = (1, 1, \ldots, 1)$ denote the vector of m ones.

Definition 1 *For* $\mathbf{a}, \mathbf{b} \in \Pi^m_{0,n}$

$$\mathbf{a} \sim \mathbf{b} \Longleftrightarrow \mathbf{a} - \mathbf{b} \equiv k\mathbf{1}_m \pmod{n+1} \text{ for some } k.$$

It is easy to see that $\sim$ is an equivalence relation and that there are $n!$ equivalence classes corresponding to the elements of $\Pi^n_{1,n}$. An alternative perspective on these permuations in terms of differences will prove to be useful.

Definition 2 *For* $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \Pi^m_{0,n}$ *let*

$$d(\mathbf{a}) = ((a_2 - a_1), (a_3 - a_2), \ldots, (a_m - a_{m-1})) \in \{1, 2, \ldots, n\}^{m-1}$$

where subtraction is modulo $n + 1$.

The following obvious lemma provides the connection to the equivalence relation.

Lemma 1 $\mathbf{a} \sim \mathbf{b} \Longleftrightarrow d(\mathbf{a}) = d(\mathbf{b})$

Lemma 2 $\mathbf{a} \in \Pi^m_{0,n}$ *if and only if* $d(\mathbf{a}) = (d_1, d_2, \ldots d_{m-1})$ *satisfies*

$$\sum_{k=i}^{j} d_k \not\equiv 0 \pmod{n+1} \text{ for } 1 \leq i \leq j \leq m-1.$$

Proof: The a_i are distinct. □

In general we will say that a string $x_1, x_2, \ldots, x_m$ of terms from $\{1, 2, \ldots, n\}$ has property $\mathcal{P}$ if all sums of consecutive terms (including a 'sum' of a single term) are distinct modulo $n + 1$. That is, if $\sum_{k=i}^{j} x_k \not\equiv 0 \pmod{n+1}$ for $1 \leq i \leq j \leq m-1$.

Denote by D_n the set of elements of $\{1, 2, \ldots n\}^{n-1}$ satisfying property $\mathcal{P}$. The one to one correspondances from Lemmas 1 and 2 between permuatations of $\{1, 2, \ldots n\}$ ($\Pi^n_{1,n}$), equivalence classes of $n-$permutations of $\{0, 1, \ldots n\}$ and length $n - 1$ vectors from $\{1, 2, \ldots, n\}$ satisfying property $\mathcal{P}$ (D_n) will be frequently used in what follows.

3 Difference Representations

Having set up the equivalence class partitions, with permutaions as representations, it is relatively straighforward to show the existance of universal cycles for D_n using standard techniques. We will need an additional property to 'lift' universal cycles for D_n to a equivalence class universal cycles for $\Pi^n_{1,n}$.

Construct the directed graph G_n with vertices corresponding to strings in $\{1, 2, \ldots, n\}^{n-2}$ satisfying property $\mathcal{P}$ and arcs corresponding to elements in D_n. The arc corresponding to $\mathbf{d} = (d_1, d_2, \ldots, d_{n-1}) \in D_n$ goes from vertex $(d_1, d_2, \ldots, d_{n-2})$ (the prefix of $\mathbf{d}$) to the vertex $(d_2, d_3, \ldots, d_{n-1})$ (the suffix of $\mathbf{d}$).

By $\mathcal{P}$, the partial sums $d_{k-1}, d_{k-2} + d_{k-1}, \ldots, d_2 + \cdots + d_{k-1}, d_1 + d_2 + \cdots d_{k-1}$ are distinct for any k. (If $j < i$ and $d_j + \cdots + d_{k-1} = d_i + \cdots + d_{k-1}$ then $d_{j+1} + \cdots + d_i = 0$ (mod $n+1$).) Thus, given $d_1, d_2, \ldots, d_k$ satisfying $\mathcal{P}$, there are $n-k$ choices for d_{k+1} so that $d_1, d_2, \ldots d_{k+1}$ satisfies $\mathcal{P}$. In particular, there are 2 choices of d_{n-1} for any prefix. That is, the outdegree of each vertex is two. (By a symmetric argument each indegree is two.) Note also that since there are $|D_n| = n!$ arcs, there are $n!/2$ vertices in G_n.

Figures 1 and 2 show D_3 and D_4. The vertices are labeled by difference prefixes/suffixes and the arcs are labeled by the corresponding element of D_n and, in italics, the permutations in $\Pi^n_{1,n}$ with this difference sequence.

The following elementary lemma shows that G_n is in general Eulerian.

Lemma 3 *G_n is strongly connected and every vertex of G_n has indegree and outdegree 2.*

Proof: The statement about the degrees follows from the discussion above.

Construct the directed graph H_n with vertices corresponding to permutations in $\Pi^{n-1}_{0,n}$ and arcs corresponding to permutations in $\Pi^n_{0,n}$. The arc $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \Pi^n_{0,n}$ goes from vertex $(a_1, a_2, \ldots, a_{n-1}) \in \Pi^{n-1}_{0,n}$ to $(a_2, a_3, \ldots, a_n) \in \Pi^{n-1}_{0,n}$. It is easy to see that the indegree and outdegree of each vertex is two. It is also not difficult to show that H_n is strongly connected (see Jackson [3].) For completeness we include a short proof of this fact.

We show how to find a path between any two arcs in H_n and thus since there are no isolated vertices, H_n is strongly connected. First note that there is a path from arc $\mathbf{x} = (x_1, x_2, \ldots x_n)$ to any cyclic permutation of $\mathbf{x}$. Namely, $(x_1, x_2, \ldots, x_n)$, $(x_2, x_3, \ldots, x_n, x_1)$, $\cdots$ $(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_{i-1})$. Since any permutation can be obtained from another by a sequence of transpositions of adjacent elements, it is enough to show that there is a path from $\mathbf{a} = (a_1, a_2, \ldots a_i, a_{i+1}, \ldots a_n)$ to $\hat{\mathbf{a}} = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n)$. Let b be the element of $\{0, 1, \ldots, n\}$ that does *not* appear in $\mathbf{a}$. Making use of paths P_1, P_2, P_3 for cyclic permutations, we have the path $\mathbf{a} = (a_1, a_2, \ldots a_i, a_{i+1}, \ldots a_n)$, P_1, $(a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1})$, $(a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1}, b)$, $(a_{i+2}, \ldots, a_n, a_1, \ldots, a_{i-1}, b, a_i)$, P_2, $(b, a_i, a_{i+2} \ldots, a_n, a_1, \ldots, a_{i-1})$, $(a_i, a_{i+2}, \ldots, a_n, a_1, \ldots, a_{i-1}, a_{i+1})$, $P_3, (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n) = \hat{\mathbf{a}}$. For example, in H_4 we have $(1, 2, 3, 4)$, $(2, 3, 4, 1)$, $(3, 4, 1, 2)$, $(4, 1, 2, 0)$, $(1, 2, 0, 3)$, $(2, 0, 3, 1)$, $(0, 3, 1, 2)$, $(3, 1, 2, 4)$, $(1, 2, 4, 3)$.

Finally, we observe that the graph obtained from H_n by identifying vertices that belong to the same equivalence class of $\Pi^{n-1}_{0,n}$ is G_n and thus G_n is strongly connected. Identify vertices in H_n corresponding to $\mathbf{a}, \mathbf{b} \in \Pi^{n-1}_{0,n}$ if $d(\mathbf{a}) = d(\mathbf{b})$. Each new vertex arises from an equivalence class of $n+1$ vertices and corresponds to a string of length $n-2$ from $\{1, 2, \ldots, n\}$ satisfying $\mathcal{P}$. That is, it corresponds to a vertex of G_n. Similarly, there is a correspondence between equivalence classes of arcs in $\Pi^n_{0,n}$ of H_n and arcs of G_n. It is not difficult to check that with these correspondences G_n is obtained from Hn. In figure 3 we give the example of H_3. $\Box$

Lemma 4 *Universal cycles for D_n exist.*

Proof: By Lemma 3, G_n is Eulerian. An Eulerian cycle produces the universal cycle. $\Box$

For example, there is one Eulerian cycle in D_3 starting with arc 11, namely 112332. One Eulerian cycle in D_4 is 111242224344431213331342.

4 Universal Cycles for $\Pi^n_{1,n}$

Finally, we need to 'lift' the universal cycles for D_n to equivalence class universal cycles for $\Pi^n_{1,n}$. Since we select a representative from each equivalence class, the cyclic representation must return to the same representative of each class. It is this 'lifting' that produces difficulties with the order isomorphic approach described in the introduction. For a given $a \in \{0, 1, \ldots, n\}$ and a universal cycle $u_1 u_2 \ldots u_{n!}$ for D_n we construct the cycle $v_1 v_2 \ldots v_{n!}$ with $u_1 = a$ and $v_i = v_{i-1} + u_{i-1}$ for $i = 2, 3, \ldots n!$, with addition modulo $n+1$. For example, with the cycle 112332 for D_3 we get

$a = 0$	012032
$a = 1$	123103
$a = 2$	230210
$a = 3$	301321

So for $a = 0$, the equivalence class representatives are $012 \sim 123$, $120 \sim 231$, $203 \sim 132$, $032 \sim 321$, $320 \sim 213$ and $201 \sim 312$.

With 111242222434444312133331342 for D_4 we get

$$\begin{array}{ll} a=0 & 012304130421042301420143 \\ a=1 & 123410241032103412031204 \\ a=2 & 234021302143214023142310 \\ a=3 & 340132413204320134203421 \\ a=4 & 401243024310431240314032 \end{array}$$

Note that in both cases, every choice of a 'lifts' to an equivalence class universal cycle. So in fact we get a family of such cycles, depending on the initial choice of a.

In general, $v_1 v_2 \dots v_{n!}$ will be cyclic if and only if $v_1 \equiv v_{n!} + u_{n!} \pmod{n+1}$. But, this is the same as $v_1 \equiv v_1 + u_1 + u_2 + \cdots u_{n!} \pmod{n+1}$ since $v_i = v_{i-1} + u_{i-1}$. So we need the following Lemma.

Lemma 5 *Let $u_1 u_2 \dots u_{n!}$ be a universal cycle for D_n. Then $\sum_{i=1}^{n!} u_i \equiv 0 \pmod{n+1}$.*

Proof: Let D_n^k denote the set of strings $d_1, \dots, d_{n-1} \in D_n$ with $d_1 = k$. That is, those strings with lead term k. For $k \in \{1, 2, \dots, n\}$, $|D_n^k| = (n-1)!$. This follows from the counts as in the description of G_n or by symmetry ($|D_n^k| = |D_n^j|$).

Let $\mathbf{d}^i = (d_1^i, d_2^i, \dots d_{n-1}^i)$ for $i = 1, 2, \dots n!$ be the list of the strings in D_n in some order. Then since each u_i is the first term of some string in D_n and conversely,

$$\begin{aligned} \sum_{i=1}^{n!} u_i &= \sum_{j=1}^{n!} d_1^j \\ &= \sum_{k=1}^{n} |D_n^k| \\ &= \sum_{k=1}^{n} (n-1)! \\ &= \frac{(n+1)n}{2}(n-1)! \\ &\equiv 0 \pmod{n+1}. \end{aligned}$$

□

Theorem 1 *There exists a complete family of equivalence class universal cycles for permutations of $\{1, 2, \dots, n\}$ using the symbols $\{0, 1, 2, \dots, n\}$.*

Proof: Immediate from the above remarks and Lemmas 4 and 5. □

Observe that by using the matrix tree theorem, counts on the number of spanning trees and hence the number of Eulerian paths in G_n can be obtained. (See for example Tutte [4].) These methods would then give a count of the numbers of equivalence class universal cycles for permutations. However, it appears that it may be difficult to obtain a general expression for the evaluation of the determinant used in these counts. As is the case with de bruijn cycles, it may be possible to use the structure of the graph G_n to obtain algorithms for generating universal cycles for permutations (and hence for generating permutations).

References

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