Multiset Poker

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CGTC45 - March 2015

Talk that would have fit this session:

2 Player Graph Pebbling Game

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Multiset Poker

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- $\binom{n}{k}$ = number of k element sets from $[n] = \{1, 2, \ldots, n\}$
- Poker Deck - 13 ranks, 4 suits for each rank $\Rightarrow$ 52 cards
- Full house - 5 card hand with 3 of one rank 2 of another

What is the probability of a full house poker hand?

- $\binom{n}{k}$ = number of k element sets from $[n] = \{1, 2, \ldots, n\}$
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What is the probability of a full house poker hand?

$$\frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}}$$

- $\left(\!\binom{n}{k}\!\right)$ = number of k element multisets from $[n]$
- Poker Deck - 13 ranks, 4 suits for each rank $\Rightarrow$ 52 cards
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What is ________________________________ ?

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- Poker Deck - 13 ranks, 4 suits for each rank $\Rightarrow$ 52 cards
- Full house - 5 card hand with 3 of one rank 2 of another

What is the probability of a full house in multiset poker?
Every 5 card multiset hand is equally likely

$$\frac{13 \cdot \left(\!\binom{4}{3}\!\right) \cdot 12 \cdot \left(\!\binom{4}{2}\!\right)}{\left(\!\binom{52}{5}\!\right)}$$

What is the probability of a full house *If we use 5 decks?*

$$\frac{13 \cdot \left(\!\binom{4}{3}\!\right) \cdot 12 \cdot \left(\!\binom{4}{2}\!\right)}{\left(\!\binom{52}{5}\!\right)}$$

What is the probability of a full house

If we use 5 decks?

NO! hands are not equally likely

$$\frac{13 \cdot \left(\!\!\binom{4}{3}\!\!\right) \cdot 12 \cdot \left(\!\!\binom{4}{2}\!\!\right)}{\left(\!\!\binom{52}{5}\!\!\right)}$$

What is the probability of a full house

If we deal with replacement?

$$\frac{13 \cdot \left(\!\!\binom{4}{3}\!\!\right) \cdot 12 \cdot \left(\!\!\binom{4}{2}\!\!\right)}{\left(\!\!\binom{52}{5}\!\!\right)}$$

What is the probability of a full house

If we deal with replacement?

NO! hands are not equally likely

$$\frac{13 \cdot \left(\!\binom{4}{3}\!\right) \cdot 12 \cdot \left(\!\binom{4}{2}\!\right)}{\left(\!\binom{52}{5}\!\right)}$$

- $\binom{[n]}{k}$ = collection of k element sets from $[n]$
- $\left(\!\!\binom{[n]}{k}\!\!\right)$ = collection of k element multisets from $[n]$

Fact

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

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Question

How can we play multiset poker with a $56 = 52 + 5 - 1$ *card deck?*

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Fact

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

Question

How can we play multiset poker with a $56 = 52 + 5 - 1$ card deck?

Add 4 cards (knights) to deck:

$C\clubsuit, C\diamondsuit, C\heartsuit, C\spadesuit$

The standard bijection:

$$\left(\!\!\binom{[6]}{5}\!\!\right) \leftrightarrow \binom{[6+5-1]}{5}$$

multiset	set	stars and bars
$1,2,3,4,5$	$1,3,5,7,9$	$\ast \mid \ast \mid \ast \mid \ast \mid \ast \mid$
$2,3,4,4,4$	$2,4,6,7,8$	$\mid \ast \mid \ast \mid \ast\ast\ast \mid\mid$
$1,1,2,3,6$	$1,2,4,6,10$	$\ast\ast \mid \ast \mid \ast \mid\mid\mid \ast$
$3,3,3,3,3$	$3,4,5,6,7$	$\mid\mid \ast\ast\ast\ast\ast \mid\mid\mid$
$2,3,4,5,6$	$2,4,6,8,10$	$\mid \ast \mid \ast \mid \ast \mid \ast \mid \ast$
$1,1,3,3,6$	$1,2,5,6,10$	$\ast\ast \mid\mid \ast\ast \mid\mid\mid \ast$

Linear order of augmented deck
Low to high ranks: $2, 3, \ldots, 9, 10, J, Q, K, A, C$
Low to high within ranks: $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$

The standard bijection

Dealt hand		multiset hand
$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$	$\Longleftrightarrow$	$3\clubsuit 3\clubsuit 3\clubsuit 3\clubsuit 3\clubsuit$
$3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$	$\Longleftrightarrow$	$3\clubsuit 3\clubsuit 4\heartsuit 10\heartsuit 10\heartsuit$
$3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit C\clubsuit$	$\Longleftrightarrow$	$3\diamondsuit 4\spadesuit 10\spadesuit 10\spadesuit A\clubsuit$
$3\diamondsuit 5\clubsuit J\heartsuit C\diamondsuit C\spadesuit$	$\Longleftrightarrow$	$3\diamondsuit 4\spadesuit J\clubsuit A\heartsuit A\spadesuit$
$3\diamondsuit 5\clubsuit J\heartsuit C\diamondsuit C\heartsuit$	$\Longleftrightarrow$	$3\diamondsuit 4\spadesuit J\clubsuit A\heartsuit A\heartsuit$

- A hand with no knight maps to itself
- Place knights in their location $C\clubsuit$, $C\diamondsuit$, $C\heartsuit$, $C\spadesuit$ $=$ 1,2,3,4 left to right
- Place remaining cards in open spaces in order
- Knights take value of first regular card to their right

The Knight's bijection

$$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit \quad \Longleftrightarrow \quad 3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$$

$$3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit \quad \Longleftrightarrow \quad 3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$$

$$C\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit \quad \Longleftrightarrow \quad 3\diamondsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$$

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Knight's bijection $C : \binom{[n+k-1]}{k} \Rightarrow \left(\!\binom{[n]}{k}\!\right)$

- For $S \in \binom{[n+k-1]}{k}$, let $T = S \cap \{n+1, n+2, \ldots, n+k-1\}$
- $|T| = t$ and $R = S \cap [n] = S - T$ with $|R| = k - t$
- Write $R = a_1 < a_2 < \cdots < a_{k-t}$ and
  $T = n + b_1 < n + b_2 < \cdots < n + b_t$
- $T' = \{b_1, b_2, \ldots, b_t\} \subset \binom{[k-1]}{t}$ is a t element set from $[k-1]$
- Use the standard bijection B to map $T' = \{b_1, b_2, \ldots, b_t\}$ to a t element multiset from $[(k-1) - t + 1] = [k - t]$
- Use these as indices of repeated elements from R.
- In particular $B(T') = \{b_i - i + 1 | i = 1, 2, \ldots, t\}$.
- Then let $R' = \{a_{b_i - i + 1} | i = 1, 2, \ldots, t\}$
- The image of S under the knight's bijection is then
  $C(S) = R \cup R'$.

Knight's bijection $C : \binom{[n+k-1]}{k} \Rightarrow \left(\!\binom{[n]}{k}\!\right)$

- Any set avoiding knights maps to itself
- Place knights in their location
- Place regular elements in order in open spots
- Knights take value of first regular element to their right

Knight's bijection $C : \binom{[n+k-1]}{k} \Rightarrow \left(\!\binom{[n]}{k}\!\right)$

- Stars and bars bijection with 'extra' elements as stars and 'regular' elements as bars

Playing poker with Knight's bijection

- No 'numerical' computations needed
- 'Normal' hands are themselves
- No 2 players can get the same card
- At most 4 instances of duplicated cards
-
- High card 'beats' one pair

General Poker Games

3 'Deals'

- *Multiple Decks (t decks)*
- *Multiset bijection*
- *Dealing with replacement*

- r ranks
- s suits
- hand size h

limit as $t \to \infty$ multideck is dealing with replacement

Notation for general poker

$\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$

Regular 13 rank poker:
$\langle 0^{11}, 1^1, 2^0, 3^0, 4^1 \rangle$ is 4 of a kind
$\langle 0^{11}, 1^0, 2^1, 3^1 \rangle$ is full house
$\langle 0^{10}, 1^2, 2^0, 3^1 \rangle$ is 3 of a kind
$\langle 0^{10}, 1^1, 2^2 \rangle$ is 2 pair
$\langle 0^9, 1^5 \rangle$ is high card

With $r = 17$ ranks and hand size $h = 9$
$\langle 0^{14}, 1^0, 2^0, 3^3 \rangle$ is three 3 of a kinds
$\langle 0^{14}, 1^1, 2^0, 3^0, 4^2 \rangle$ is two 4 of a kinds

Observe that full house and 4 of a kind have same exponents as do 2 pair and 3 of a kind
$r = \sum p_i$ and $h = \sum i \cdot p_i$

Definition

The number of poker hands of type λ is

$$N(\lambda) = N_{ra}(\lambda) \cdot N_{su}(\lambda)$$

$\#$ ways to pick the ranks $\cdot$ $\#$ ways to pick the suits
Rank selection $N_{ra}(\lambda)$ does not depend on deal type

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Rank selection $N_{ra}(\lambda)$ *does not depend on deal type*

Fact

For a poker hand of type $\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$, *the number of ways to pick the ranks is the multinomial coefficient*

$$N_{ra} = \binom{r}{p_0, p_1, p_2, \ldots}$$

Does not depend on method of dealing

4 of a kind and full house:
$N_{ra}(\langle 0^{11}, 1^1, 2^0, 3^0, 4^1 \rangle) = N_{ra}(\langle 0^{11}, 1^0, 2^1, 3^1 \rangle)$

$$= \binom{13}{11, 1, 1} = \frac{13!}{11!1!1!} = 13 \cdot 12$$

3 of a kind and 2 pair:
$N_{ra}(\langle 0^{10}, 1^2, 2^0, 3^1 \rangle) = N_{ra}(\langle 0^{10}, 1^1, 2^2 \rangle)$

$$= \binom{13}{10, 2, 1} = \frac{13!}{10!2!1!} = \frac{13 \cdot 12 \cdot 11}{2}$$

two four of a kind with $r = 17$ ranks and hand size $h = 9$:

$$N_{ra}(\langle 0^{14}, 1^1, 2^0, 3^0, 4^2 \rangle) = \binom{17}{14, 2, 1} = \frac{17!}{14!2!1!}$$

Definition

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$\#$ ways to pick the ranks $\cdot$ $\#$ ways to pick the suits
Suit selection $N_{ra}(\lambda)$ does depend on deal type

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Fact

For a poker hand of type $\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$, the number of ways to pick the suits is

- *(t decks):* $N_{su}^{md}(\lambda) = \prod \binom{st}{i}^{p_i}$
- *(multiset):* $N_{su}^{ms}(\lambda) = \prod \left(\!\binom{s}{i}\!\right)^{p_i}$
- *(dealing with replacement):* $N_s^r(\lambda) = \binom{h}{\lambda} \cdot s^h$

Regular poker full house probabilities (including full house flush)
$\langle 0^{11}, 1^{0}, 2^{1}, 3^{1} \rangle$

5 decks:

$$\frac{\binom{13}{11,1,1} \cdot \binom{20}{3}\binom{20}{2}}{\binom{260}{5}}$$

multiset:

$$\frac{\binom{13}{11,1,1} \cdot \left(\!\binom{4}{3}\!\right) \left(\!\binom{4}{2}\!\right)}{\left(\!\binom{52}{5}\!\right)}$$

Dealing with replacement:

$$\frac{\binom{13}{11,1,1} \cdot \binom{5}{3,2} \cdot 4^5}{52^5}$$

17 ranks, 3 suits, 9 card hands
two 4 of a kind (including flushes), $\langle 0^{14}, 1^1, 2^0, 3^0, 4^2 \rangle$

2 decks:

$$\frac{\binom{17}{14,2,1} \cdot \binom{6}{4}^2 \binom{6}{1}}{\binom{102}{9}}$$

multiset:

$$\frac{\binom{17}{14,2,1} \cdot \left(\!\!\binom{3}{4}\!\!\right)^2 \left(\!\!\binom{3}{1}\!\!\right)}{\left(\!\!\binom{51}{9}\!\!\right)}$$

Dealing with replacement:

$$\frac{\binom{17}{14,2,1} \cdot \binom{9}{4,4,1} \cdot 3^9}{51^9}$$

$\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$

t decks

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \prod \binom{st}{i}^{p_i}}{\binom{rst}{h}}$$

Multiset

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \prod \left(\!\binom{s}{i}\!\right)^{p_i}}{\left(\!\binom{rs}{h}\!\right)}$$

Dealing with replacement

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \binom{h}{(0!)^{p_0}\,(1!)^{p_1}\,(2!)^{p_2}\,(3!)^{p_3}\ldots} \cdot s^h}{(rs)^h}$$

Multiset vs. regular probabilities (as percents %)

	multiset	regular
Straight flush	.001	.001
5 kind flush	.001	0
4 kind flush	.016	0
full house flush	.016	0
5 kind	.02	0
3 kind flush	.09	0
2 pair flush	.09	0
flush	.13	.20
straight	.27	.39
pair flush	.30	0
4 kind	.56	.02
full house	.80	.14
3 kind	7.10	2.87
2 pair	8.90	4.75
High card	34.10	49.68
1 pair	47.62	42.3

regular poker hands $\binom{52}{5} = 2,598,960$

multiset poker hands $\left(\!\binom{52}{5}\!\right) = \binom{56}{5} = 3,819,816$

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