Degree Lists for 2 Two tree like graph classes

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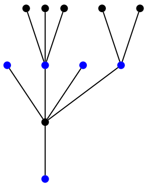
Recall degree list conditions for trees
A basic exercise in a first graph theory course

- Degrees are positive integers and degree sum is even (always assume this)
- Trees are bipartite
  $\Rightarrow$ degrees partition into two parts with equal sum
- Trees (on n vertices) have $n-1$ edges
  $\Rightarrow$ Degree sum is $2n-2$



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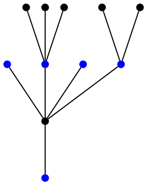
Positive integers $d_1, d_2, \ldots, d_n$ are degrees of a tree $\Leftrightarrow$
$\sum d_i = 2n-2$



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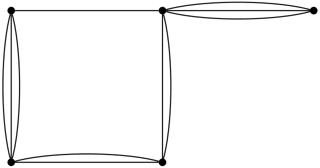
Recall degree list conditions for loopless multigraphs
another basic exercise in a first graph theory course

- Degrees are positive integers and degree sum is even
- No loops
  $\Rightarrow$ edges from max degree vertex go to other vertices
  $\Rightarrow$ max degree $\leq$ sum of other degrees



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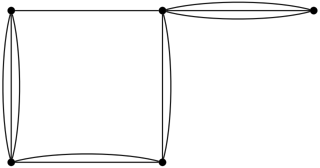
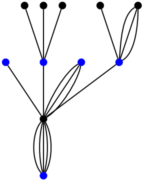


Positive integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with even degree sum, are degrees of a loopless multigraph $\Leftrightarrow d_1 \leq \sum_{i=2}^{n} d_i$

Loopless multitree



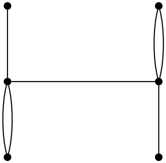
Exercise - What are conditions for degree lists of multitrees?

Degree conditions for multitrees?

Positive integers $d_1, d_2, \ldots, d_n$ are degrees of a multiforest
$\Leftrightarrow$ degrees partition into two parts with equal sum

- easy exercise(s), induction; switching, ...
- Get $d_1 \leq \sum_{i=1}^{n} d_i$ and even degree sum for free
- Note that partitioning integer lists into equal sum parts is NP-hard
- Need a little more for multitrees

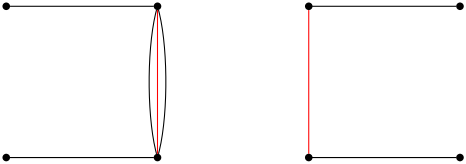
In a multiforest:
If all d_i are even then edge multiplicities are all even



- 'Proof': parity argument
- In general edge multiplicities are multiplies of $gcd(d_1, , \dots, d_n)$

Positive integers $d_1, d_2, \ldots, d_n$ that partition into two parts with equal sum realize a multitree if $\dfrac{\sum d_i}{gcd} \geq 2n-2$

Get multiforest and the use switching to get multitree



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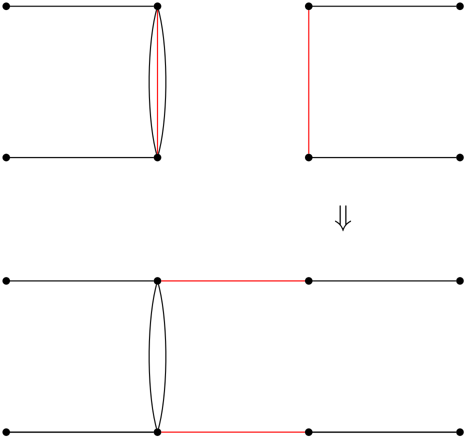

It would have been nice to have the multigraph condition instead of the NP-hard partition condition

Non-Theorem: Positive integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with even degree sum, are degrees of a loopless multitree $\Leftrightarrow d_1 \leq \sum_{i=2}^{n} d_i$
Moreover can be realized with a maximum degree vertex adjacent to a minimum degree vertex

False Proof by example:
Case 1: $d_1 = d_2$: $7, 7, \ldots, 5, 3$
By induction construct multitree $4, 7, \ldots, 5, 3$
add 3 edges between first vertex and a new degree 3 vertex

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False Proof by example:
Case 2: $d_1 > d_2$: $9, 7, \ldots, 5, 3$
By induction construct multitree $8, 7, \ldots, 5, 2$ with edges between degree 8 and degree 2 add a(nother) edge between 8 and 2

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False Proof by example:

Basis?
Case 1 collapses at $n = 3$

- multitree degree conditions equivalent to NP-hard problem
- So look at 2-multitrees (each edge multiplicity 1 or 2)
- Should not expect anything nice for k-multitrees as partition is NP-hard

2-multiforest conditions, $d_1 \geq \ldots, \geq d_n$ with even degree sum

- If all d_i even $\Rightarrow$ edge multiplicities all 2 $\Rightarrow \frac{d_1}{2}, \frac{d_2}{2}, \ldots, \frac{d_n}{2}$ are degrees of a forest
  i.e., sum is a multiple of 4 and at most $2(2n-2) = 4n-4$
- At most 2 edges to each vertex $\Rightarrow d_1 \leq 2(n-1)$
- At least 2 'leaves' $\Rightarrow$ at least two d_i are 1 or 2
- At most $2(n-1)$ edges $\Rightarrow$ degree sum at most $4n-4$

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- Each odd degree vertex adjacent to edge with multiplicity 1
  $\Rightarrow$ degree sum $\leq 4n - 4 - \#$odd degrees
- Remove degree 1 vertices
  $\Rightarrow$ what is left can't have too large a degree sum
  $\Rightarrow$ degree sum $\leq 4n - 4 - 2 \cdot (\#\text{degree 1 vertices})$

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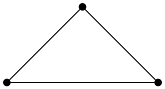
Conditions are also sufficient

- Connected unless all d_i even and $\sum d_i < 4n-4$ or some d_i odd and $\sum d_i < 2n-2$
- Get partition into parts with equal sum for 'free'

2-trees

'Build' by repeatedly attaching a 'pendent' vertex to an edge



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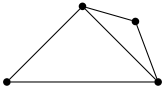

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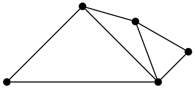

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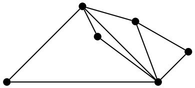

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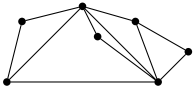

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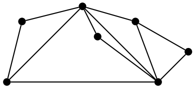

Necessary Conditions for degrees of a 2-tree

- degree sum is $4n - 6$
- $n - 1 \geq d_1 \geq \ldots \geq d_n \geq 2$
- There are at least two $d_i = 2$
- list is not $\left\langle \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, 2, 2, \ldots, 2 \right\rangle$
- All d_i even $\Rightarrow (\#\ d_i = 2) \geq \frac{n+3}{3}$

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Theorem (Bose, Dujmovic, Kriznac, Langerman, Morin, Wood, Wuher 2008)

Necessary and sufficient for degree lists of 2-trees

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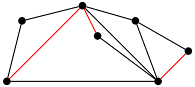
Necessary and sufficient for degree lists of 2-trees

- If some d_i is odd 'almost always' works if degree sum is $4n - 6$
- If all d_i even need 'about' 1/3 of the d_i to be 2

Partial 2-tree: subgraph of a 2-tree



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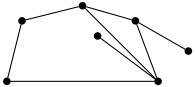

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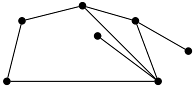

- K_4 minor free graphs
- series-parallel graphs
  construct: add pendent edge; replace edge with a path, add parallel edges

Necessary conditions for degrees of a partial 2-tree

g is the number of 'missing' edges $\Rightarrow \sum d_I = 4n - 6 - 2g$

- When $g = 0$ list is not $\langle \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, 2, 2, \ldots, 2 \rangle$
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Theorem

Necessary and sufficient for degree lists of partial 2-trees

- When some d_i is odd condition is essentially $(\# \ d_i = 1) \leq g$
- If all d_i even $(\# \ d_i = 2) \geq \frac{n+3-2g}{3}$ holds whenever $\sum d_i \leq \frac{18}{5}(n - 1)$

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Proof: Earlier false proof with corrected basis

Degrees of a loopless multigraph can be realized as a unicylclic multitree (unless $\frac{\sum d_i}{gcd}$ is too small)

Proof: Adjust 2 of the d_i to get partition into equal sums
construct a multitree
then add multiedges between 2 adjusted vertices