Degree lists for multforests and near multiforests

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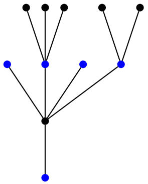
Recall degree list conditions for trees
A basic exercise in a first graph theory course

- Degrees are positive integers and degree sum is even (always assume this)
- Trees (on n vertices) have $n-1$ edges
  $\Rightarrow$ Degree sum is $2n-2$

Positive integers $d_1, d_2, \ldots, d_n$ are degrees of a tree $\Leftrightarrow$
$\sum d_i = 2n-2$

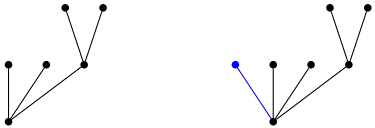


$(5, 4, 3, 1, 1, 1, 1, 1, 1, 1, 1)$

(One) proof of

Positive integers $d_1, d_2, \ldots, d_n$ are degrees of a tree $\Leftrightarrow$
$\sum d_i = 2n - 2$

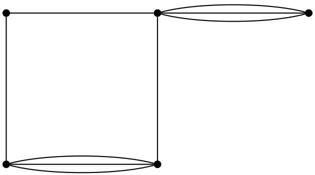
- $d_1 \geq \cdots \geq d_{n-1} \geq d_n$ with $\sum d_i = 2n - 2$
  $\Rightarrow d_n = 1$ and $d_1 \geq 2$
- By induction, tree with $d_1 - 1, d_2, \ldots, d_{n-1}$
- Add edge $v_1 v_n$



$(4, 3, 1, 1, 1, 1, 1) \Rightarrow \; (3, 3, 1, 1, 1, 1, \quad) \Rightarrow \; (4, 3, 1, 1, 1, 1, 1)$

Recall degree list conditions for loopless multigraphs
another basic exercise in a first graph theory course

- Degrees are positive integers and degree sum is even
- No loops
  $\Rightarrow$ edges from max degree vertex go to other vertices
  $\Rightarrow$ max degree $\leq$ sum of other degrees



Positive integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with even degree sum, are degrees of a loopless multigraph $\Leftrightarrow$ $d_1 \leq \sum_{i=2}^{n} d_i$

(one) proof of

Positive integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with even degree sum, are degrees of a loopless multigraph $\Leftrightarrow$ $d_1 \leq \sum_{i=2}^{n} d_i$

- $d_1 \leq d_2 + \cdots + d_n \Rightarrow d_1 - d_n \leq d_2 + \cdots + d_{n-1}$
- $d_2 \leq d_1$ and $d_n \leq d_{n-1} \Rightarrow d_2 \leq (d_1 - d_n) + d_3 + \cdots + d_{n-1}$
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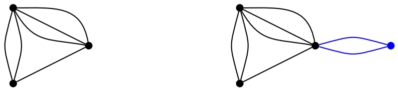
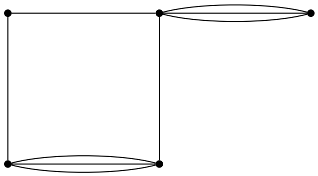


$(6, 5, 3, 2) \Rightarrow \quad (4, 5, 3, \quad) \Rightarrow \quad (6, 5, 3, 2)$

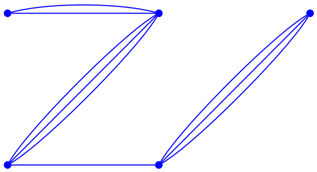
Both proofs added a 'leaf' $\Rightarrow$ no cycles created

So: Have we just proved?

Non-Theorem: *Positive integers* $d_1 \geq d_2 \geq \cdots \geq d_n$ *with even degree sum, are degrees of a loopless multitree* $\Leftrightarrow d_1 \leq \sum_{i=2}^{n} d_i$
i.e. Multigraph $\Rightarrow$ *Multitree with same degrees*



$(5, 4, 4, 3, 2)$



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- $(2, 2, 2, 2)$? not connected, so forest not tree
- Forests are bipartite so $d_1 \leq d_2 + \cdots d_n \Rightarrow$ can partition d_i into two parts with equal sum
- Above fails for $(3, 4, 5)$
- Test if given integer list partitions into 2 equal sum parts? NP-hard problem so something is wrong

What went wrong with multgraph proof?

Positive integers $d_1 \geq d_2 \geq \cdots \geq d_n$ *with even degree sum, are degrees of a loopless multigraph* $\Leftrightarrow$ $d_1 \leq \sum_{i=2}^{n} d_i$

- $d_1 \leq d_2 + \cdots + d_n \Rightarrow d_1 - d_n \leq d_2 + \cdots + d_{n-1}$
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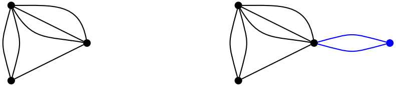
- Add edges $v_1 v_n$



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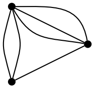
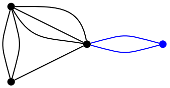
- $d_2 \leq d_1$ and $d_n \leq d_{n-1} \Rightarrow d_2 \leq (d_1 - d_n) + d_3 + \cdots + d_{n-1}$

IF $n \geq 4$

- By induction multigraph with $d_1 - d_n, d_2, \ldots, d_{n-1}$
- Add edges $v_1 v_n$





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With correct basis for $n = 3$ we get

Degrees of a multigraph $d_1 \leq d_2 + \cdots + d_n$ have a realization with underlying graph a forest or a graph with exactly one cycle (which is a triangle)

Note that partitioning integer lists into equal sum parts is NP-hard. So might not detect forest realization if there is one.

Good example why need basis for induction

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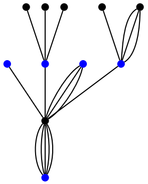
Good example why need basis for induction

- What are conditions for a multiforest?
- What if we want connected? i.e., multitree?

Loopless multitree

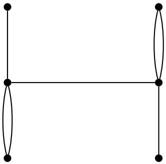


Degree conditions for multitrees?

Positive integers $d_1, d_2, \ldots, d_n$ are degrees of a multiforest
$\Leftrightarrow$ degrees partition into two parts with equal sum

- easy exercise(s), induction; switching, ...
- Get $d_1 \leq \sum_{i=1}^{n} d_i$ and even degree sum for free
- Need a little more for (connected) multitrees

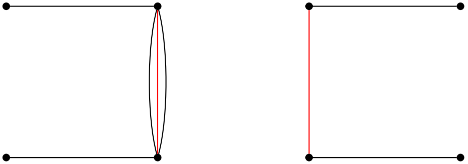
In a multiforest:
If all d_i are even then edge multiplicities are all even



- 'Proof': parity argument
- In general edge multiplicities are multiples of $gcd(d_1, , \ldots, d_n)$

Positive integers $d_1, d_2, \ldots, d_n$ that partition into two parts with equal sum realize a multitree if $\dfrac{\sum d_i}{gcd} \geq 2n - 2$

Get multiforest and the use switching to get multitree



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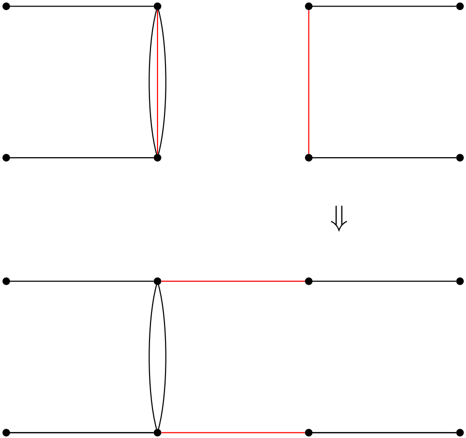
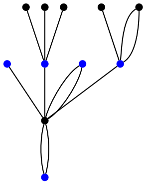


- multitree degree conditions equivalent to NP-hard problem
- So look at 2-multitrees (each edge multiplicity 1 or 2)
- Should not expect anything nice for k-multitrees as partition is NP-hard

2-multitree



2-multiforest conditions, $d_1 \geq \ldots, \geq d_n$ with even degree sum

- If all d_i even $\Rightarrow$ edge multiplicities all 2 $\Rightarrow \frac{d_1}{2}, \frac{d_2}{2}, \ldots, \frac{d_n}{2}$ are degrees of a forest
  i.e., sum is a multiple of 4 and at most $2(2n-2) = 4n-4$

- At most 2 edges to each vertex $\Rightarrow d_1 \leq 2(n-1)$
- At least 2 'leaves' $\Rightarrow$ at least two d_i are 1 or 2
- At most $2(n-1)$ edges $\Rightarrow$ degree sum at most $4n-4$

Last 3 will be implied by further conditions

More 2-multiforest conditions

- Each odd degree vertex adjacent to edge with multiplicity 1
  $\Rightarrow$ degree sum $\leq 4n - 4 - \#$odd degrees
- Remove degree 1 vertices
  $\Rightarrow$ what is left can't have too large a degree sum
  $\Rightarrow$ degree sum $\leq 4n - 4 - 2 \cdot (\#$degree 1 vertices$)$

Conditions are also sufficient

Positive integers $d_1, d_2, \ldots, d_n$ with even degree sum are degrees of a 2-multiforest $\Leftrightarrow$

- *When all d_i even: $\sum d_i \leq 4n - 4$ and a multiple of 4*
- *Some d_i odd: $\sum d_i \leq 4n - 4 - \max\{n_{odd}, 2n_1\}$*

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Positive integers $d_1, d_2, \ldots, d_n$ with even degree sum are degrees of a 2-multiforest $\Leftrightarrow$

- *When all d_i even:* $\sum d_i \leq 4n - 4$ *and a multiple of 4*
- *Some d_i odd:* $\sum d_i \leq 4n - 4 - \max\{n_{odd}, 2n_1\}$

Proof: exercise

All proofs in this talk (probably) could be assigned as exercises in a beginning undergraduate discrete math or graph theory course